Method of Cylindrical Shells is used when it becomes complicated to compute inner and outer radii of a washer. For example in Figure 1, we must solve for *x* in terms of *y*.

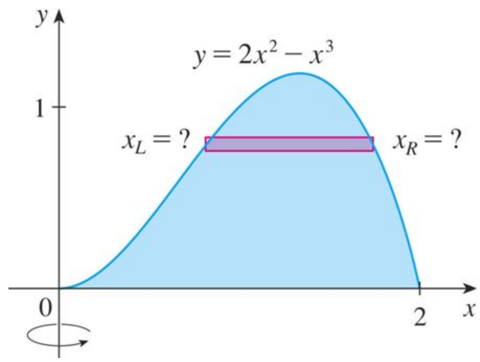

Figure 1

Figure 2 shows one cylindrical shell with inner radius r_1, outer radius r_2, and height h.

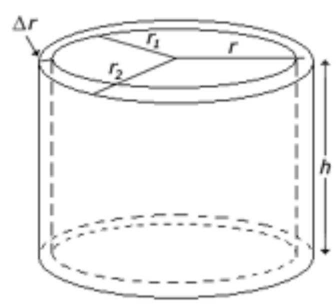

Figure 2

Its volume V is calculated as follows $V = V_2 - V_1$, where V_1 is a volume of the cylinder with inner radius r_1 and V_2 is a volume of the cylinder with outer radius r_2. Remember that the volume of a cylinder is $V = \pi r^2 h$. So, we have the following formula for the volume of one cylinder

$$V = \pi {r_2}^2 h - \pi {r_1}^2 h = \pi({r_2}^2 - {r_1}^2)h = \pi(r_2 + r_1)(r_2 - r_1)h$$

Observe that $r = \frac{r_2+r_1}{2}$. So by multiplying and dividing the previous equation by 2 we get

$V = 2\pi \, \frac{r_2+r_1}{2} \, h(r_2 + r_1)(r_2 - r_1)$

$$\boldsymbol{V = 2\pi r h \Delta r}$$

Now let S be the solid obtained by rotating about y-axis the region bounded by $y = f(x)$ where $f(x) \geq 0, y = 0, x = a,$ and $x = b,$ where $b > a \geq 0$.

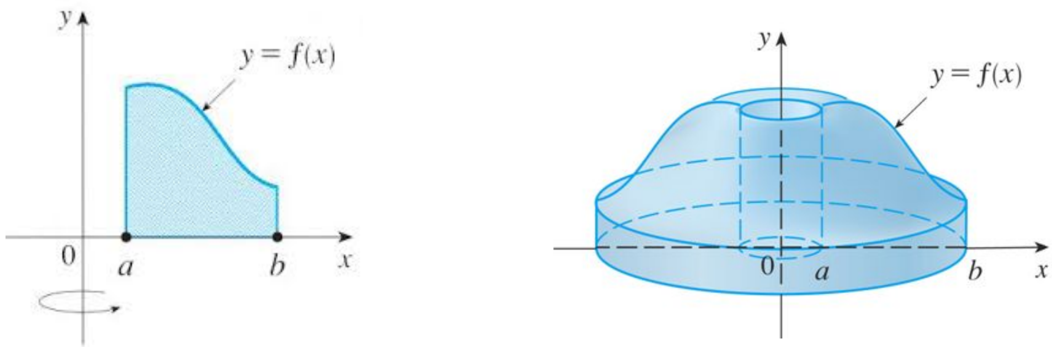

Figure 3

To obtain this volume, we divide the interval [*a, b*] into n subintervals $[x_{i-1}, x_i]$ of equal width Δx and let $\bar{x_i}$ be the midpoint of the *i*-th subinterval.

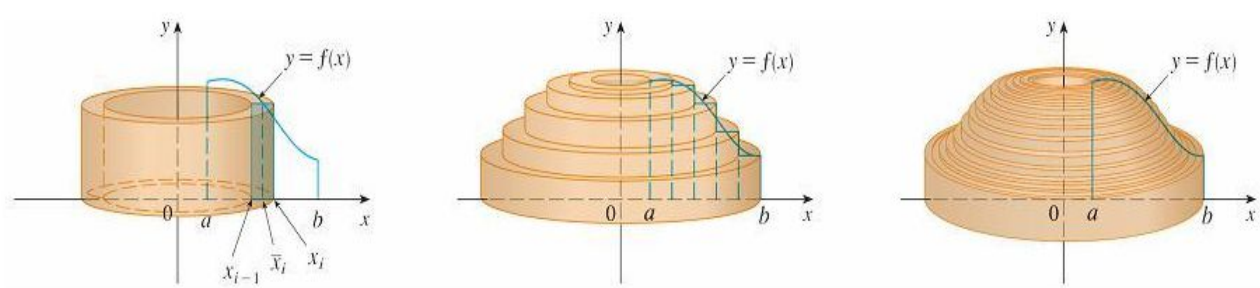

Figure 4

Then the volume of the solid in Figure 3, obtained by rotating about the y-axis the region under the curve $y = f(x)$ from *a* to *b*, is

$$\boldsymbol{V = \int_a^b 2\pi x f(x) dx \qquad where\ 0 \leq a < b}$$

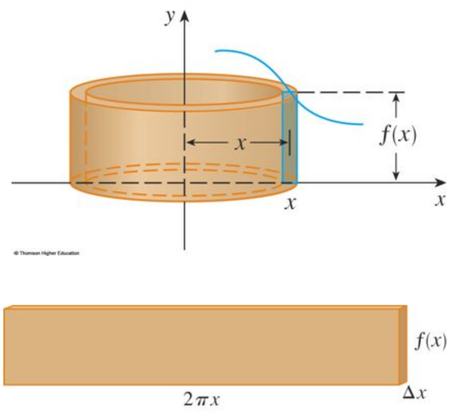

Figure 5

In Figure 5, we cut and flattened a typical cylindrical shell. So we have

$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \quad \underbrace{[f(x)]}_{\text{height}} \quad \underbrace{dx}_{\text{thickness}}$$

Observe that $\boldsymbol{x}$ represents the radius of each cylinder when we rotate the region under a curve about the y-axis. If we revolve about the x-axis, then the radius is $\boldsymbol{y}$.

Example 1 Find the volume of the solid obtained by rotating about the y-axis the region between $y = x \text{ and } y = x^2$

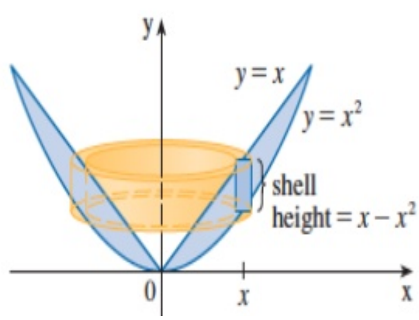

Figure 6

Solution:

The shell has radius $\boldsymbol{x}$, circumference $\boldsymbol{2\pi x}$, and the height $\boldsymbol{x - x^2}$. So, the volume is

$$V = \int_0^1 (2\pi x)(x - x^2)dx = 2\pi \int_0^1 (x^2 - x^3)dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}$$

Example 2 Find the volume of the solid obtained by rotating the region bounded by

$y = x - x^2$ and $y = 0$ about the line $x = 2$.

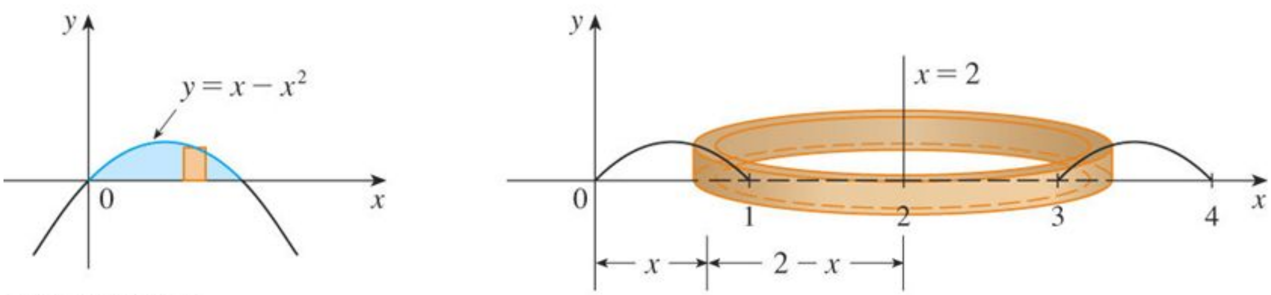

Figure 7

Solution:

Figure 7 shows the region and cylindrical shell formed by rotation about the line $x = 2$. It has radius $2 - x$, circumference $2\pi(2 - x)$, and height $x - x^2$.

The volume of the given solid is

$$V = \int_0^1 2\pi(2 - x)(x - x^2)\, dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x) dx = 2\pi \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \frac{\pi}{2}$$