A. The degree of the numerator is *greater than* the degree of the denominator.

1) Perform long division.
2) Integrate each term.

Example:

$$\int \frac{x^3+x}{x-1}dx$$

After performing long division and integration, we get

$$\int \frac{x^3+x}{x-1}dx = \int (x^2+x+2+\frac{2}{x-1})dx = \frac{x^3}{3}+\frac{x^2}{2}+2x+2ln|x-1|+C$$

B. The degree of the numerator is *less than* the degree of the denominator.

Let's consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. Then

Factor in denominator $\boldsymbol{Q(x)}$			Term in partial fraction decomposition
1	Product of distinct linear factors	$(a_1x+b_1)(a_2x+b_2)\ldots(a_kx+b_k)$	$\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_k}{a_kx+b_k}$
2	Product of linear factors, some of which are repeated	$(ax+b)^k$	$\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots+\frac{A_k}{(a_1x+b_1)^k}$
3	Irreducible quadratic factors, none of which is repeated	ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$

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Factor in denominator $Q(x)$			**Term in partial fraction decomposition**
4	Repeated irreducible quadratic factor	$(ax^2+bx+c)^k$	$\frac{A_1+B_1}{ax^2+bx+c}+\frac{A_2+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_2+B_k}{(ax^2+bx+c)^k}$

Examples:

1. **Product of distinct linear factors.**

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$$

Factor the denominator.

$$2x^3+3x^2-2x=x(2x^2+3x-2)=x(2x-1)(x+2)$$

$$\frac{x^2+2x-1}{2x^3+3x^2-2x}=\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}$$

To determine the values of A, B, and C, we multiply both sides of the equation by the product of the denominators, $x(2x-1)(x+2)$, obtaining

$$\begin{aligned}x^2+2x-1&=A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)\\&=A(2x^2+3x-2)+Bx^2+2Bx+2Cx^2-Cx\\&=2Ax^2+3Ax-2A+Bx^2+2Bx+2Cx^2-Cx\\&=(2A+B+2C)x^2+(3A+2B-C)x-2A\end{aligned}$$

Since the left hand side and the right hand side of the equation are equal, so coefficients must be equal. Thus,

$2A+B+2C=1, 3A+2B-C=2$, and $-2A=-1$

Solving, we get $A=\frac{1}{2}, B=\frac{1}{5}, C=-\frac{1}{10}$, and so

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx = \int \left[\frac{1}{2}\left(\frac{1}{x}\right)+\frac{1}{5}\left(\frac{1}{2x-1}\right)-\frac{1}{10}\left(\frac{1}{x+2}\right)\right]dx=\frac{1}{2}ln|x|+\frac{1}{10}ln|2x-1| -\frac{1}{10}ln|x+2|+K$$

2. **Product of linear factors, some of which are repeated.**

$$\int \frac{4x}{x^3 - x^2 - x + 1} dx$$

Factor the denominator.

$$x^3 - x^2 - x + 1 = x^2(x-1) - (x-1) = (x^2-1)(x-1) = (x-1)(x+1)(x-1)$$
$$= (x-1)^2(x+1)$$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2 = (A+C)x^2 + (B-2C)x + (-A+B+C)$$

Comparing coefficients, we get

$A + C = 0$, $B - 2C = 4$, and $-A + B + C = 0$

Solving, we obtain $A = 1, B = 2, C = -1$, so

$$\int \frac{4x}{x^3 - x^2 - x + 1} dx = \int \frac{4x}{(x-1)^2(x+1)} dx = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\right) dx$$
$$= ln|x-1| - \frac{2}{x-1} - ln|x+1| + K = ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + K$$

3. **Irreducible quadratic factors, none of which is repeated.**

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$$

Since $x^3 + 4x = x(x^2+4)$ can't be factored further, we write

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)x = (A+B)x^2 + Cx + 4A$$

So, $A + B = 2, C = -1, 4A = 4$, thus $A = 1, B = 1, C = -1$ and thus

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$$
$$= \int \frac{2x^2 - x + 4}{x(x^2+4)} dx$$
$$= \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right) dx = \int \frac{1}{x} dx + \int \frac{x}{x^2+4} dx - \int \frac{1}{x^2+4} dx$$

$$= ln|x| + \frac{1}{2} ln(x^2+4) - \frac{1}{2} tan^{-1}\left(\frac{x}{2}\right) + K$$

4. Repeated irreducible quadratic factor.

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2} dx$$

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\begin{aligned} -x^3+2x^2-x+1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A \end{aligned}$$

Comparing coefficients, we get $A+B=0, C=-1, 2A+B+D=2, C+E=-1, A=1$, then $A=1, B=-1, C=-1, D=1, E=0$. Thus

$$\begin{aligned} \int \frac{1-x+2x^2-x^3}{x(x^2+1)^2} dx \\ &= \int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right) dx \\ &= \int \frac{1}{x} dx - \int \frac{x}{x^2+1} dx - \int \frac{1}{x^2+1} dx + \int \frac{x}{(x^2+1)^2} dx \\ &= ln|x| - \frac{1}{2} ln(x^2+1) - tan^{-1}x - \frac{1}{2(x^2+1)} + K \end{aligned}$$