Working with Polar Curves

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Tangent Lines of Polar Curves

Recall that $x = r\cos\theta$ and $y = r\sin\theta$. Also, note that r is a function of θ. With this information we can find the slope of a tangent line of a polar curve.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r(\theta)\cos\theta + r'(\theta)\sin\theta}{r(\theta)\sin\theta + r'(\theta)\cos\theta}$$

Area Between Polar Curves

When finding the area between polar curves we can use the following:

$$A = \int_{\alpha} \frac{1}{2}(r_{outer}^2 \quad r_{inner}^2)d\theta$$

Note that α and are angles since our integral is with respect to θ. Also, r_{outer} and r_{inner} are functions of θ.

In the case that we want r_{inner} to be the origin, we set it equal to 0. Thus the equation becomes:

$$A = \int_{\alpha} \frac{1}{2}(r_{outer}^2)d\theta \qquad \text{OR} \qquad A = \int_{\alpha} \frac{1}{2}r^2 d\theta$$

Examples

Let's apply these to some examples. Consider the circle shape on page 2 where $r = 2\cos\theta$.

At which angles does the slope of the tangent line equal 0 for $0 \leq \theta \leq \pi$?

$$\frac{dy}{dx} = \frac{2\cos\theta \cdot \cos\theta \quad 2\sin\theta \cdot \sin\theta}{2\cos\theta \cdot \sin\theta \quad 2\sin\theta \cdot \cos\theta} = \frac{2\cos(2\theta)}{2\sin(2\theta)} = \cot 2\theta$$

If we let $0 = \frac{dy}{dx} = \cot(2\theta)$ we see θ must equal $\frac{\pi}{4}$ or $\frac{3\pi}{4}$.

What is the area between r and the origin for $0 \leq \theta \leq \frac{\pi}{2}$?

After some simplifying and use of the half angle formula we get the following:

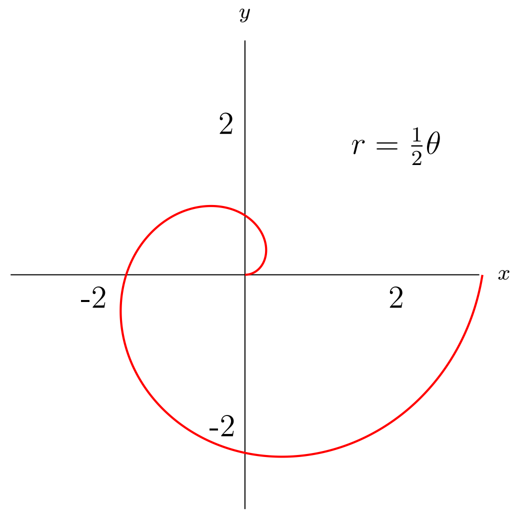
$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2}(2\cos(2\theta))^2 d\theta = \int_0^{\frac{\pi}{2}} 2\cos^2(2\theta)d\theta = \int_0^{\frac{\pi}{2}} 2 \cdot \frac{1+\cos(4\theta)}{2}d\theta$$

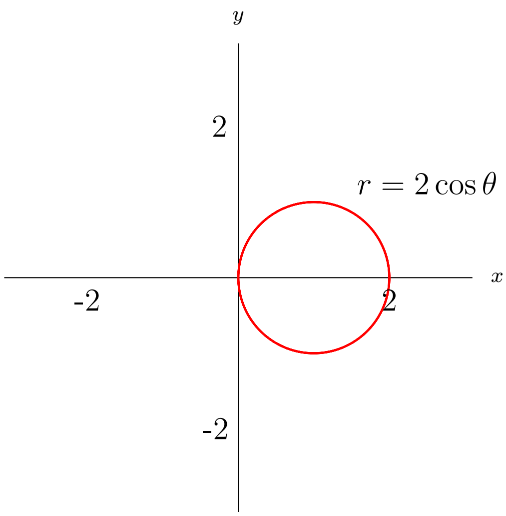
$$= [\theta + \frac{\sin 4\theta}{4}]_0^{\frac{\pi}{2}} = \frac{\pi}{2} + 0 \quad (0+0) = \frac{\pi}{2}$$

There are many different shapes when constructing polar curves. This handout includes some examples.

Archimedean Spiral

To the right is an archimedean spiral. Equations of the form $r = n\theta$, where $n \in \mathbb{R}$, create this shape.



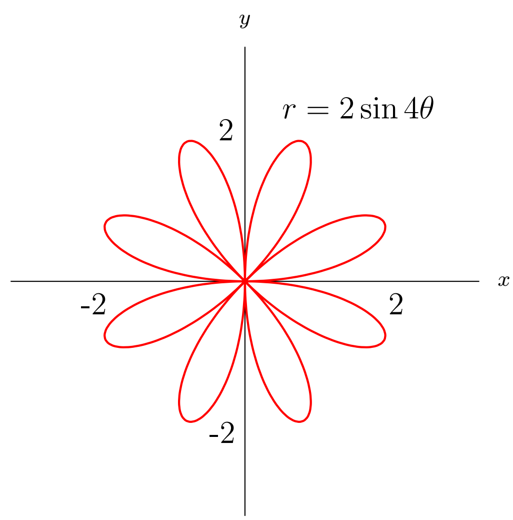


Circles

Equations of the form $r = a\sin\theta$ or $r = a\cos\theta$, for $a \in \mathbb{R}$, create circular curves on the axes. Note, that an equation of the form $r = a$ would create a circle of radius a, centered at the origin. Here we have the $\cos\theta$ case. If it were $\sin\theta$, the circle would be on the y-axis.

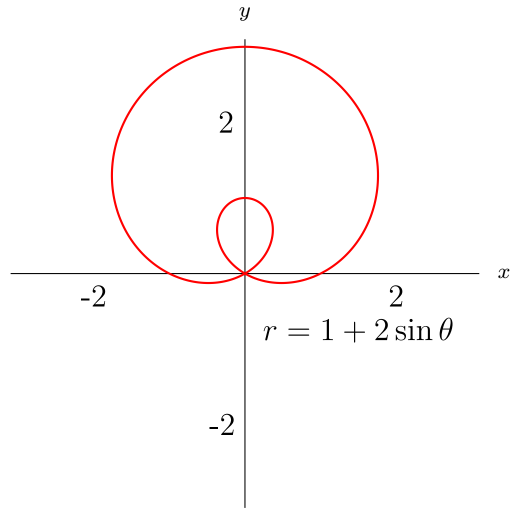
Roses

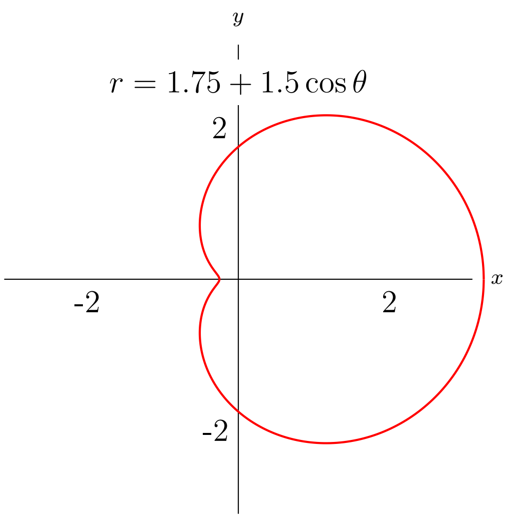
An equation of $r = a\cos n\theta$ or $r = a\sin n\theta$, for $a, n \in \mathbb{R}$, results in rose curves. When n is an odd value, there are n petals on the curve. If n is even, then there are $2n$ petals.



We next look at equations with the generic form of $r = b + a\sin\theta$ and $r = b + a\cos\theta$, where a and b are nonzero real numbers. These curves are called **limacons**.

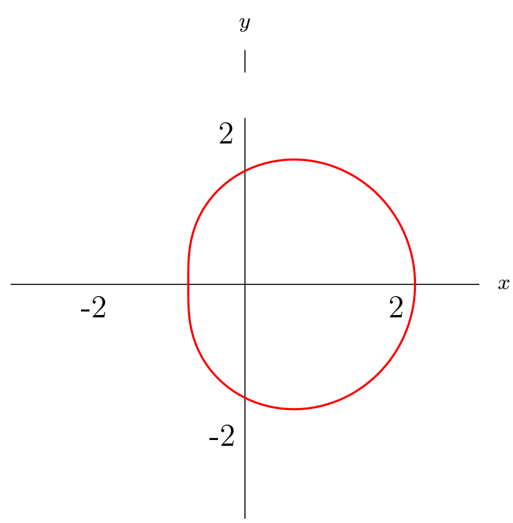
Limacon (Inner and Outer Loop)
If $b < a$, the curve has an inner and outer loop such as the graph to the right. Note that if we had $\cos\theta$ instead, the curve would be symmetric about the x-axis. A change in the sign of a would flip the shape in both cases. A change in the sign of b does not affect the shape.

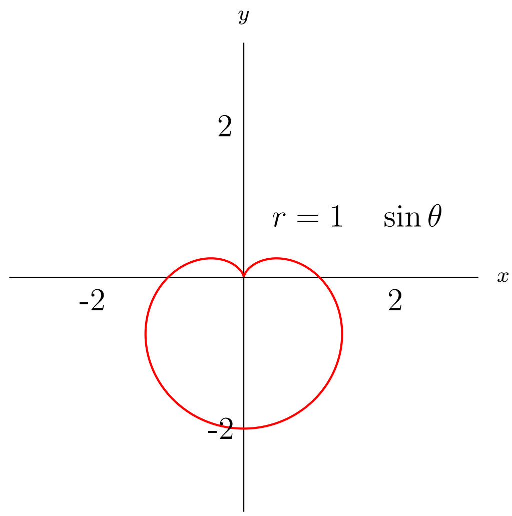




Dimpled Limacon
If $a < b < 2a$, the curve is a dimpled limacon. The rules for symmetry and orientation are the same as our previous shape.

Convex Limacon
If $b \quad 2a$, the curve is a convex limacon. The same rules apply as the previous limacons.





Cardioid
If $a = b$, we get a special limacon called a cardioid. The same rules apply as the previous shapes. The cardioid looks similar to the dimpled limacon, however the cardioid's "dimple" is sharper while the dimpled limacon is smoother.

The final shape we will discuss is the **lemniscate**.

Lemniscate
An equation resulting in this shape has the general form of $r^2 = a^2 \sin 2\theta$ or $r^2 = a^2 \cos 2\theta$. The lemniscate looks like an infinity symbol or a figure 8 depending on if we have $\sin 2\theta$ or $\cos 2\theta$. a determines the size of the graph.