Disk/Washer Method Handout

The disk/washer method is one of the more difficult methods in Calculus 2 according to popular opinion. This is due to having to use past mathematical knowledge and imagination to figure out the volume of an object that rotates around an axis.

Definition: Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \int_a^b A(x)dx$$

Procedure:

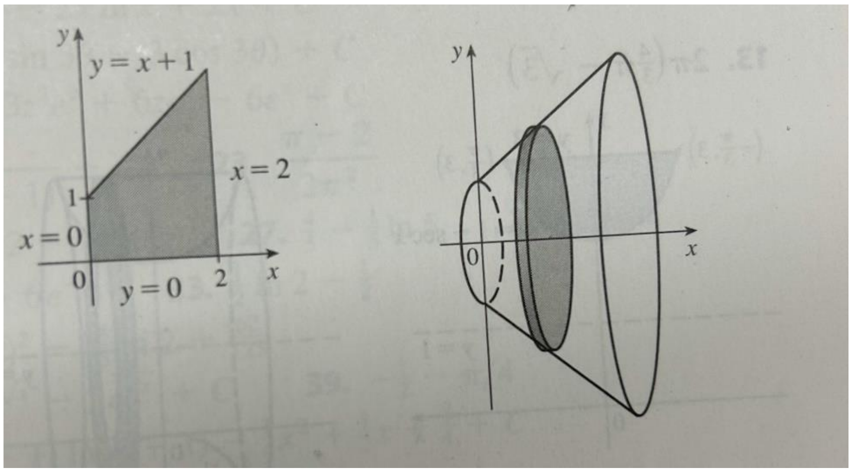
1) Draw a graph based on the criteria you are given (Use your knowledge of how the graph might look depending on the formula given). Draw what the rotation around the given axis would look like.
2) Determine the bounds of the integral by finding the point(s) of intersection between the two lines. If the graph rotates around the y-axis, rewrite the equation such that x is by itself. If the graph rotates around the x-axis, rewrite the equation such that y is by itself.
3) Determine the radius/radii.
If the rotating region turns out to be a disk: The radius should be the line given where x is a function of y (or y is a function of x).
If the rotating region turns out to be a washer: There will be two radii – with one line being the outer radius and the other line being the inner radius depending on the graph.
If the region rotates around a point other than the origin (0,0): Add or subtract that value to the original radius
4) If the rotating region is a disk, plug in the radius to πr^2 (the area of a circle). If the rotating region is a washer, plug in both radii into πr^2 separately, then do $\pi(outer\ radius)^2 - \pi(inner\ radius)^2$.
5) Integrate what you get for step 4 with the bounds you found in step 2. Integrate in respect to the variable used in step 2.

If the procedure is confusing, hopefully some examples will clear things up:

Example: Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

$y = x + 1,\ y = 0,\ x = 0,\ x = 2$; about the x-axis

1) Draw a graph based on the criteria you are given (Use your knowledge of how the graph might look depending on the formula given). Draw what the rotation around the given axis would look like.



On the left, $y = x + 1$, $x = 0$, $x = 2$, and $y = 0$ are all graphed, creating the shaded region as shown. On the right, a picture is drawn displaying what the rotation about the x-axis would look like.

2) Determine the bounds of the integral by finding the point(s) of intersection between the two lines. If the graph rotates around the y-axis, rewrite the equation such that x is by itself. If the graph rotates around the x-axis, rewrite the equation such that y is by itself.

The graph rotates about the x-axis, so we will keep $y = x + 1$ as it is, and we will integrate with respect to x. On the integral, our bounds will be from 0 to 2 since we are integrating with respect to x and the region extends from 0 to 2 on the x-axis.

3) Determine the radius/radii.

Our rotating region turns out to be a disk, and the radius of the disk in this case is half the length of the disk, which is $y = x + 1$, or we can just say $x + 1$.

4) If the rotating region is a disk, plug in the radius to πr^2 (the area of a circle).

We know our radius, $x + 1$, so we will plug $x + 1$ into πr^2, which becomes

$$\pi(x+1)^2 = \pi(x+1)(x+1) = (\pi x + \pi)(x+1) = \pi x^2 + 2\pi x + \pi$$

(We simplify so our equation will be easier to integrate in step 5.)

5) Integrate what you get for step 4 with the bounds you found in step 2. Integrate in respect to the variable used in step 2.

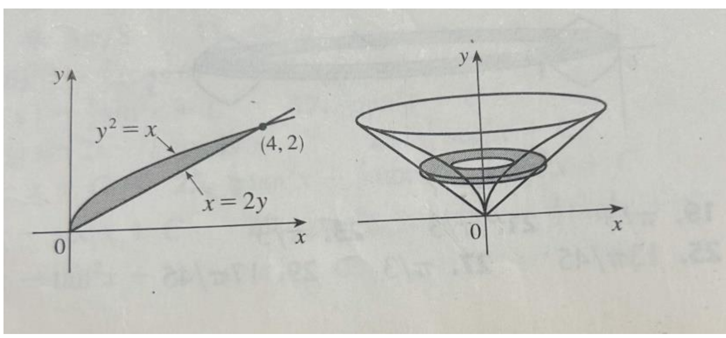
In step 2, we found our bounds to be 0 and 2, and our integration in respect to the variable x. Our integration is as follows:

$$\int_0^2 (\pi x^2 + 2\pi x + \pi)\, dx$$

Using our knowledge of integration, we will end up with a volume of $\frac{26}{3}\pi$.

Example: $y^2 = x,\ x = 2y$; about the y-axis

1) Draw a graph based on the criteria you are given (Use your knowledge of how the graph might look depending on the formula given). Draw what the rotation around the given axis would look like.



On the left, $y^2 = x$ and $x = 2y$ are graphed, created the shaded region as shown. On the right, the shaded region's rotation around the y-axis is shown. The drawing turns out to be a washer.

2) Determine the bounds of the integral by finding the point(s) of intersection between the two lines. If the graph rotates around the y-axis, rewrite the equation such that x is by itself. If the graph rotates around the x-axis, rewrite the equation such that y is by itself.

Since the graph rotates around the y-axis, we will leave the equations as they are since x is already a function of y in both. We can find the bounds of the integral by setting $y^2 = 2y$. Dividing y on both sides, we get 2 as an intersection point. It turns out that 0 is a solution as well. So, our bounds for the integral will be 0 and 2.

3) Determine the radius/radii.

Since we have a washer, there will be two radii, with the outer radius being $2y$ and the inner radius being y^2.

4) If the rotating region is a washer, plug in both radii into πr^2 separately, then do $\pi(outer\ radius)^2 - \pi(inner\ radius)^2$.

Plugging in our outer radius and inner radius, we have $\pi(2y)^2 - \pi(y^2)^2 = 4\pi y^2 - \pi y^4$

(Like the previous example, simplifying makes the integration process a little easier)

5) Integrate what you get for step 4 with the bounds you found in step 2. Integrate in respect to the variable used in step 2.

From step 2, we got 0 and 2 as our bounds and we will be integrating in respect to y:

$$\int_0^2 (4\pi y^2 - \pi y^4)dy$$

Using our knowledge of integration, we end up getting $\frac{64\pi}{15}$ as our volume.