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A Reversible Nearest Particle System on the Homogeneous Tree[1]

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We introduce a modified contact process on the homogeneous tree. The modification is to the death rate: an occupied site becomes vacant at rate one if the number of occupied id neighbors is at most one. This modification leads to a growth model which is reversible, off the empty set, provided the initial set of occupied sites is connected. Reversibility admits tools for studying the survival properties of the system not available in a nonreversible situation. Four potential phases are considered: extinction, weak survival, strong survival, and complete convergence. The main result of this paper is that there is exactly one phase transition on the binary tree. Furthermore, the value of the birth parameter at which the phase transition occurs is explicitly computed In particulars survival and complete convergence hold if the birth parameter exceeds 1/4. Otherwise, the expected extinction time is finite.

KEY WORDS: Trees; growth models; phase transition; reversible; flows.

1. INTRODUCTION

The (single parameter) uniform model η_t is a continuous time Markov process taking values in $X = \{0, 1\}^{\mathbb{T}^d}$, where $\mathbb{T}^d$ denotes the homogeneous tree in which each vertex has degree $d+1$. An element η of X is referred to as a configuration and the value of η at the site x denoted by $\eta(x)$ is the spin at x. If the spin at x is 1, we say that the site x is occupied. Otherwise, x is vacant. The evolution of the process is that a vacant site becomes occupied at a rate which is proportional to the number of occupied neighbors, while an occupied site becomes vacant at rate one if at most one

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of its neighbors is occupied. More formally, the flip rate $c(x, \eta)$ at site $x \in \mathbb{T}^d$ in configuration η is given by

$$c(x, \eta) = \begin{cases} \beta \sum_{\|x-y\|=1} \eta(y) & \text{if } \eta(x) = 0 \\ 1 & \text{if } \eta(x) = 1 \text{ and } \sum_{\|x-y\|=1} \eta(y) \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\beta > 0$ and $\|x - y\|$ denotes the length of the shortest path connecting x and y. See Liggett[5] [Ch. I, Sect. 3] for a complete construction of the process.

Some may recognize these dynamics as a modification of the contact process where the rate at which an occupied site becomes vacant is one regardless of the spin values in the neighborhood. The effect of the modification is that connected components remain connected until absorption in the empty set. Furthermore, the configuration with all sites occupied is absorbing so that $\delta_{\mathbf{1}}$ is the upper invariant measure for the process, where **1** is the configuration in which all sites are occupied. Another distinction is that the finite system is reversible with respect to the measure $\pi(A) = \beta^{|A|}$ for all finite, connected $A \subset \mathbb{T}^d$, where we have identified a configuration η in X with the subset A of the vertices of $\mathbb{T}^d$ which are occupied. The contact process properties of additivity and self-duality fail for the uniform model, while attractiveness is preserved.

Liggett[6] first introduced the two parameter version of this process in 1985. It has both an interior birth rate λ and an exterior birth rate $\gamma < 1/d$. Given a configuration η, let $\mathscr{G}(\eta)$ be the minimal connected subgraph of $\mathbb{T}^d$ containing η. The rate at which a vacant site becomes occupied in configuration η decays exponentially with the distance to $\mathscr{G}(\eta)$, while occupied sites become vacant at rate one. The flip rates are given by

$$c(x, \eta) = \lambda \gamma^{\|x - \mathscr{G}(\eta)\|}(1 - \eta(x)) + \eta(x)$$

where $\|x - \mathscr{G}(\eta)\| = \min\{\|x - y\| : y \in \mathscr{G}(\eta)\}$. The two parameter model is reversible with respect to the measure $\mu(A) = \gamma^{|\mathscr{G}(A)|} \lambda^{|A|}$ for finite $A \subset \mathbb{T}^d$. Liggett studied the survival properties of the finite system and gave bounds on the critical value of the interior birth parameter in terms of the exterior birth parameter. The connection between the single and double parameter models is that the single parameter uniform model can be regarded as a limit of the double parameter version. To see this, set the double parameter nearest neighbor birth rate $\lambda\gamma$ constantly equal to β while letting the exterior birth rate γ tend to zero. In particular, the rate at which vacant sites at a distance strictly greater than one from $\mathscr{G}(\eta)$ become occupied tends to zero. Since the interior birth rate $\lambda = \beta/\gamma$, the interior birth rate tends to infinity. Thus, any occupied site in the interior of $\mathscr{G}(\eta)$ which becomes vacant is instantaneously reoccupied.

In the study of reversible interacting particle systems, new tools become available which, in some cases, allow more complete analysis. Attractive Reversible Nearest Particle Systems (RNPS) on $\mathbb{Z}$ provide a successful example. In a Nearest Particle System (NPS), an occupied site becomes vacant at rate one and a vacant site becomes occupied at a rate which depends on the distances to the nearest occupied sites. The contact process provides a (nonreversible) example of a NPS in which the rate at which a vacant site becomes occupied is proportional to the number of occupied sites within distance one. On $\mathbb{Z}$, reversibility is equivalent to the assumption that the rate at which a vacant site becomes occupied takes the form

$$\frac{\beta(l_x(\eta))\,\beta(r_x(\eta))}{\beta(l_x(\eta)+r_x(\eta))}$$

for some function $\beta\colon \mathbb{N} \to \mathbb{R}_+$ such that $\sum_{l=1}^{\infty} \beta(l) < \infty$ where $l_x(\eta)$ (resp. $r_x(\eta)$) denotes the distance to the nearest occupied site to the left (resp. right) of x in configuration η. In contrast to the contact process on $\mathbb{Z}$, critical values for both the finite and infinite RNPS can be computed exactly and the upper invariant measure is the well understood stationary renewal measure whose increments are determined by the function $\beta(\cdot)$. See Liggett[5] [Ch. VII] for a full discussion of NPS.

The theory of RNPS on graphs other than $\mathbb{Z}$ is not well developed. Two important obstacles prevent generalization. Firstly, on what quantity should the rate of occupancy depend; that is, how should one generalize the notion of the nearest particle to the left and right? Secondly, there is no generalization of a renewal measure even to $\mathbb{Z}^d$ for $d \geqslant 2$. Liggett introduced the two parameter uniform model in order to extend the theory of RNPS to $\mathbb{T}^d$. Some other attempts to study RNPS on graphs besides $\mathbb{Z}$ include Chen[1,2] and Liggett.[8]

Our goal in this paper is to introduce the single parameter uniform model and to exploit reversibility to provide a complete analysis like that available for RNPS. Motivated by the contact process on $\mathbb{T}^d$, we consider the following critical values of the birth parameter. Let τ denote the time of absorption into the empty set. Let O be a distinguished vertex referred to as the origin, or the root.

$$\beta_1(d) = \sup\{\beta : \mathbb{E}^O(\tau) < \infty\}$$
$$\beta_2(d) = \inf\{\beta : P^O(\eta_t \neq \emptyset \ \forall t) > 0\}$$
$$\beta_3(d) = \inf\{\beta : P^O(O \in \eta_t \text{ for unbounded } t) > 0\}$$
$$\beta_4(d) = \inf\{\beta > \beta_3(d) : \text{for all finite, connected } A \subset \mathbb{T}^d$$
$$P^A(\eta_t \in \cdot) \to P^A(\eta_t \neq \emptyset \ \forall t)\, \delta_{\mathbf{1}}(\cdot) + P^A(\eta_t = \emptyset \text{ some } t)\, \delta_{\mathbf{0}}(\cdot)\}$$

It is immediate that $\beta_1(d) \leqslant \beta_2(d) \leqslant \beta_3(d) \leqslant \beta_4(d)$. If the process is not absorbed into the empty set in finite time we say that the process survives globally. Thus, $\beta_2(d)$ denotes the global survival threshold. If the process occupies the origin at an unbounded sequence of times, then we say that the process survives locally so that $\beta_3(d)$ denotes the local survival threshold. Global survival without local survival is weak survival. In particular, weak survival occurs with positive probability if $\beta \in (\beta_2(d), \beta_3(d))$. If $P^A(\eta_t \in \cdot) \to P^A(\eta_t \neq \emptyset \ \forall t)\, \delta_{\mathbf{1}}(\cdot) + P^A(\eta_t = \emptyset \text{ some } t)\, \delta_{\mathbf{0}}(\cdot)$ for all finite, connected A, we say that complete convergence holds. It is immediate that complete convergence holds for $\beta < \beta_2(d)$. In the local survival phase, it is not obvious that complete convergence is a monotone increasing property of the birth parameter and therefore that the definition of $\beta_4(d)$ is useful. The fact that $\mathbb{T}^d$ is a homogeneous graph implies the desired monotonicity as will be shown in Section 2.

Theorem 1 summarizes the main results proved in this paper regarding critical values for the (single parameter) uniform model. On the binary tree, all critical values are computed exactly paralleling the analysis of RNPS on $\mathbb{Z}$.

Theorem 1. (a) For $d \geqslant 2$, $\beta_2(d) = \beta_3(d)$.

(b) For $d \geqslant 2$,

$$\beta_1(d) = \frac{1}{d}\left(\frac{d-1}{d}\right)^{d-1}$$

Furthermore, at $\beta_1(d)$ the expected extinction time is finite.

(c) For $d \geqslant 2$,

$$\beta_4(d) \leqslant \frac{d}{2(d-1)^2}$$

(d) For $d = 2$, $\beta_1(2) = \beta_4(2) = \frac{1}{4}$.

Similar results for the contact process on $\mathbb{T}^d$ include $\beta_1(d) \leqslant \beta_2(d) < \beta_3(d) = \beta_4(d)$ for all $d \geqslant 2$ [see Liggett;[7] Pemantle;[10] Stacey;[14] and

Zhang[15]]. Theorem 1a states that in contrast to the contact process on $\mathbb{T}^d$ the uniform model has no intermediate phase characterized by weak survival for all $d \geq 2$. This is proved in Section 2. By (b), $\beta_1(d)$ is asymptotically $1/ed$ and the bound given in (c) is asymptotically $1/2d$. Parts (b) and (c) are proved in Sections 4 and 6.2 respectively. In Section 5, we show that $\beta_4(d) \leq 1/d$, which is better than (c) only when $d \leq 3$. The point here is that the $1/d$ bound is the analog of the upper bound that Liggett obtained for the double parameter model and that it is easily obtained. Part (d) states that in fact there is no intermediate phase in $d=2$ and identifies the exact location of the phase transition. This is proved in Section 6.3.

The technique used to push the upper bound on $\beta_4(2)$ down to $\beta_1(2)$ may work for general d. The remaining obstacle is to show that a certain set of equations has a solution which is absolutely bounded by one. A limiting version of these equations yields a partial differential equation. This PDE does in fact have a solution which is absolutely bounded by one.

Theorem 2. For $d \geq 3$, let $\alpha^*: \mathbb{R}_+^d \to \mathbb{R}$ be defined by

$$\alpha^*(x_1,..., x_d) = \sum_{i=2}^{d} \frac{(x_1 - x_i)(x_1^2 + 10x_1x_i + x_i^2)(x_2 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_d)}{d(d-2)(x_1 + \cdots + x_d)(x_1 + x_i)^3} + \frac{1}{d}$$

Then $\alpha^*(x_1,..., x_d)$ is symmetric in the variables $x_2,..., x_d$, absolutely bounded by one, and a solution to

$$\sum_{i=1}^{d} \alpha^*(x_i, x_1,..., x_{i-1}, x_{i+1},..., x_d) = 1$$

$$\sum_{i=1}^{d} \left(\frac{3}{2x_i} - \frac{\partial}{\partial x_i}\right) \alpha^*(x_i, x_1,..., x_{i-1}, x_{i+1},..., x_d) = \frac{3}{2(x_1 + \cdots + x_d)}$$

The analysis of the PDE which is presented in Section 7 may be of independent interest. Firstly, the PDE relates values of the function and its derivatives at distinct (not necessarily close) points in the positive quadrant. Furthermore, simple inspection of the PDE does not suggest a particular form for a candidate solution. Therefore, some strategy must be implemented in order to find the solution exhibited in Theorem 2.

Theorem 2 suggests that the next conjecture holds. The conjecture implies that the uniform model undergoes exactly one phase transition on all homogeneous trees.

Conjecture 1. For $d \geq 3$, $\beta_4(d) = (1/d)((d-1)/d)^{d-1}$.

2. THE SURVIVAL PHASE

In this section, we study the model when $\beta > \beta_2(d)$. As a consequence of connectedness and attractiveness, it turns out that $\beta_2(d) = \beta_3(d)$. Hence, there is no intermediate phase which is characterized by weak survival. As a consequence of attractiveness and homogeneity of the tree, weak survival does not occur above the local survival threshold. Combining these two statements, if the process survives, then it survives locally. Since the upper invariant measure is simply δ_1, homogeneity of the graph and attractiveness also imply that survival together with complete convergence is a monotone increasing property of the birth parameter.

First, Theorem 1a is proved. Then, techniques used by Salzano and Schonmann[12] for the contact process are applied to show that weak survival does not occur above the local survival threshold for the uniform model. Finally, survival together with complete convergence is equated to a property which is immediately recognizable as monotone increasing in β.

Proof of Theorem 1a. It suffices to show that $P(\eta_t^O \neq \varnothing\ \forall t) > 0$ implies that $P(O \in \eta_t^O$ for unbounded $t) > 0$. Let $\mathbb{B}_i^d = \{x \in \mathbb{T}^d : \|x - x_i\| \leqslant \|O - x\|\} \cup O$ where $x_1,..., x_{d+1}$ denote the $d+1$ nearest neighbors of the root O. By rotational symmetry,

$$P(\mathbb{B}_i^d \cap \eta_t^O \neq \varnothing) \geqslant \frac{P(\eta_t^O \neq \varnothing)}{d+1} \geqslant \frac{P(\eta_s^O \neq \varnothing\ \forall s)}{d+1}$$

Using the fact that the uniform model is an attractive spin system and that δ_O is positively correlated [see Liggett,[5] Ch. II, Thm. 2.14],

$$\begin{aligned} &P(\mathbb{B}_i^d \cap \eta_t^O \neq \varnothing, \mathbb{B}_j^d \cap \eta_t^O \neq \varnothing) \\ &\qquad \geqslant P(\mathbb{B}_i^d \cap \eta_t^O \neq \varnothing)\, P(\mathbb{B}_j^d \cap \eta_t^O \neq \varnothing), \qquad i \neq j \end{aligned}$$

Since η_t^O is connected, $P(O \in \eta_t^O) \geqslant P(\mathbb{B}_i^d \cap \eta_t^O \neq \varnothing, \mathbb{B}_j^d \cap \eta_t^O \neq \varnothing)$. Therefore, the assumption that $P(\eta_s^O \neq \varnothing\ \forall s) > 0$ implies that $P(O \in \eta_t^O)$ is bounded away from 0. Hence, $P(O \in \eta_s^O$ for unbounded $s) > 0)$. □

Remark 1. A slight modification of this proof works for the double parameter model. There connectedness of the single parameter model is replaced by connectedness of $\mathscr{G}(\eta)$.

Salzano and Schonmann[12] proved that weak survival does not occur for the contact process on homogeneous graphs in the local survival phase. The properties of the contact process which their proof uses are that it is translation invariant, strong Markov, and attractive. Therefore, the probability of weak survival is zero above the local survival threshold for any translation invariant, attractive strong Markov process on a homogeneous graph G taking values in $\{0, 1\}^G$. In particular, when $\beta > \beta_3(d)$

$$P(\eta_t^A \neq \emptyset \ \forall t) = P(O \in \eta_t^A \text{ for unbounded } t) \tag{2.1}$$

for any finite initial configuration A.

Here is the main idea behind their proof. Let X_t be an attractive, strong Markov process taking values in $\{0, 1\}^G$. They make the observation that local survival is almost surely equivalent to the event that for every $n \in \mathbb{N}$ there exists a finite time T_n such that the process contains a (fully occupied) ball of radius n centered at the origin. Using this fact, they prove that $P(O \in X_t^A$ for unbounded $t) > 0$, implies that

$$\lim_{n \to \infty} P(O \in X_t^{B(O,n)} \text{ for unbounded } t) = 1 \tag{2.2}$$

where $B(O, n)$ denotes the ball of radius n centered at the origin. On the event that the process survives, a ball of size n must become occupied somewhere. By the strong Markov property, the process can be restarted at this random time. Homogeneity of the graph and (2.2) imply that the probability of weak survival tends to zero as n tends to infinity. For a complete proof, see Salzano and Schonmann,[12] Thm. 2(i). These ideas also lead to a proof of Lemma 1.

Lemma 1. For $\beta > \beta_2(d)$, complete convergence holds if and only if

$$\lim_{n \to \infty} \liminf_{t \to \infty} P(O \in \eta_t^{B(O,n)}) = 1$$

In particular, if $P(\eta_t^O \neq \emptyset \ \forall t) > 0$ and complete convergence holds at β^*, then the same is true for all $\beta > \beta^*$.

Proof. First assume that $P(\eta_t^O \neq \emptyset \ \forall t) > 0$ and that complete convergence holds. Since the upper invariant measure is δ_1, $\lim_{t \to \infty} P(O \in \eta_t^{B(O,n)}) = P(\eta_t^{B(O,n)} \neq \emptyset \ \forall t)$. This together with (2.1) and (2.2), gives the if direction of the implication.

Assuming that $\lim_{n \to \infty} \liminf_{t \to \infty} P(O \in \eta_t^{B(O,n)}) = 1$, it is immediate that $P(\eta_t^O \neq \emptyset \ \forall t) > 0$. Given finite $A \subset \mathbb{T}^d$, let $T_n = \inf\{t : B(O, n) \subseteq \eta_t^A\}$. For $s < t$,

$$\begin{aligned} P(O \in \eta_t^A) &\geqslant P(O \in \eta_t^A \mid T_n \leqslant s)\, P(T_n \leqslant s) \\ &\geqslant \inf_{t-s \leqslant u} P(O \in \eta_u^{B(O,n)})\, P(T_n \leqslant s) \end{aligned}$$

where the final inequality follows from the strong Markov property and attractiveness. Therefore, for all $s \in \mathbb{R}_+$ and $n \in \mathbb{N}$,

$$\liminf_{t \to \infty} P(O \in \eta_t^A) \geqslant \liminf_{t \to \infty} P(O \in \eta_t^{B(O,n)})\, P(T_n \leqslant s)$$

Recall the Salzano and Schonmann[12] observation that $\lim_{n \to \infty} \lim_{s \to \infty} P(T_n \leqslant s) = P(O \in \eta_t^A$ for unbounded $t)$. This together with (2.1) implies that

$$\liminf_{t \to \infty} P(O \in \eta_t^A) \geqslant P(\eta_t^A \neq \varnothing\ \forall t)$$

Since $P(O \in \eta_t^A) \leqslant P(\eta_s^A \neq \varnothing\ \forall s \leqslant t)$, it follows that $\limsup_{t \to \infty} P(O \in \eta_t^A) \leqslant P(\eta_t^A \neq \varnothing\ \forall t)$. Thus,

$$\lim_{t \to \infty} P(O \in \eta_t^A \mid \eta_t^A \neq \varnothing\ \forall t) = 1$$

It follows that for all finite $B \subset \mathbb{T}^d$, $\lim_{t \to \infty} P(B \subseteq \eta_t^A \mid \eta_t^A \neq \varnothing\ \forall t) = 1$ which completes the proof. $\square$

3. REDUCTION TO A SINGLE BRANCH

It will be convenient to analyze the behavior of the uniform model on a single branch $\mathbb{B}^d$ of $\mathbb{T}^d$. Recall that $\mathbb{B}_i^d = \{x \in \mathbb{T}^d : \|x - x_i\| \leqslant \|O - x\|\} \cup O$ where $x_1, ..., x_{d+1}$ denote the $d+1$ nearest neighbors of the root O. Take $\mathbb{B}^d = \mathbb{B}_1^d$ and consider the initial configuration $\eta_0 = (\mathbb{T}^d \backslash \mathbb{B}^d) \cup O$. By connectedness, $\eta_t \supseteq \eta_0$ for all $t \geqslant 0$. Therefore, it suffices to keep track of the intersection with $\mathbb{B}^d$, namely $A_t = \eta_t \cap \mathbb{B}^d$. The Markov chain A_t is irreducible with state space $\mathscr{C}^d = \{$finite, connected $A \subset \mathbb{B}^d$ containing $O\}$ and rates $q(A, B)$. For $A \in \mathscr{C}^d$, say that $x \in A$ is a leaf if $x \neq O$ and $|\{y \in A : \|x - y\| = 1\}| = 1$. Denote the set of all vertices in A which are leaves by ∂A. Make the convention that the cardinality of A is the number of vertices in $A \backslash O$, i.e., $|A| = |\{x : x \in A \backslash O\}|$. Since

$$\pi(A)\, q(A, A \cup \{x\}) = \beta^{|A|+1} = \pi(A \cup \{x\})\, q(A \cup \{x\}, A)$$

for all $x \in \mathbb{B}^d$ such that $\|x - A\| = 1$, A_t is reversible with respect to the measure $\pi(A) = \beta^{|A|}$. The connection between the behavior of the finite interacting particle system and the Markov chain A_t is outlined in Theorem 3.

Theorem 3. (a) If $(A_t)_{t \geq 0}$ is positive recurrent, then $\mathbb{E}^O(\tau) < \infty$.

(b) If $(A_t)_{t \geq 0}$ is transient, then $\beta \geq \beta_4(d)$.

Proof. Let ξ_t denote the product of $d+1$ independent copies of A_t with initial state $\{O\}$. Paste together the $d+1$ roots, one on top of the other, and locate the roots at the origin of $\mathbb{T}^d$. By this correspondence, the product chain is equal in distribution to a uniform model on $\mathbb{T}^d$ with death at O suppressed. Let η_t^O denote the uniform model on $\mathbb{T}^d$ with initial state O. By attractiveness, we can couple η_t^O and ξ_t such that

$$\eta_t^O \subseteq \xi_t, \qquad \forall t \geq 0 \tag{3.1}$$

Furthermore, for any initial configuration A containing O we can couple η_t^A and ξ_t such that

$$\xi_t \subseteq \eta_t^A \qquad \forall 0 \leq t < R \tag{3.2}$$

where $R = \inf\{t : O \notin \eta_t^A\}$ (see Liggett,[5] Ch. III, Sect. 1).

The positive recurrence of A_t is equivalent to positive recurrence of ξ_t. Let $T_0 = 0$. For $i \geq 1$, set $T'_{i-1} = \inf\{t > T_{i-1} : \xi_t \neq \{O\}\}$ and $T_i = \inf\{t > T'_{i-1} : \xi_t = \{O\}\}$. Thus T_i denotes the time at which the product chain makes its ith visit to $\{O\}$. Let $N = \min\{n : \eta_{T_n}^O = \varnothing\}$. By the strong Markov property and (3.1), N is geometric with parameter $p = P(\eta_{T_1}^O = \varnothing)$. Therefore,

$$\mathbb{E}^O(\tau) \leq \mathbb{E}(T_N) = \mathbb{E}\left(\sum_{i=1}^{N} (T_i - T_{i-1})\right) = \mathbb{E}(N)\, \mathbb{E}(T_1)$$

where the final equality is an application of Wald's Lemma, establishing (a).

Assume A_t is transient. As before, x_i $i = 1,..., d+1$ denote the $d+1$ nearest neighbors of the origin. Let $S = \inf\{s : \xi_t \supseteq \{O, x_1,..., x_{d+1}\}$ for all $t \geq s\}$. Since A_t is transient, $P(S < \infty) = 1$. By (3.2),

$$\begin{aligned} P(O \in \eta_t^A \text{ for all } t \geq 0) &= P(O \in \eta_t^A \text{ for all } t \leq S) \\ &\geq P(O \in \eta_t^A \text{ for all } t \leq S \mid S \leq u)\, P(S \leq u) \\ &= P(O \in \eta_t^A \text{ for all } t \leq u \mid S \leq u)\, P(S \leq u) \end{aligned}$$

for any initial configuration A containing O and $u > 0$ such that $P(S \leq u) > 0$. Since $\{O \in \eta_t^A$ for all $t \leq u\}$ and $\{S \leq u\}$ are increasing events, η_t is an attractive spin system, and δ_A is positively correlated,

$$P(O \in \eta_t^A \text{ for all } t \leq u \mid S \leq u) \geq P(O \in \eta_t^A \text{ for all } t \leq u)$$

(see Liggett,[5] Ch. II, Cor. 2.21). Thus,

$$P(O \in \eta_t^A \text{ for all } t \geqslant 0) \geqslant P(O \in \eta_t^A \text{ for all } t \leqslant u)\, P(S \leqslant u) \tag{3.3}$$

By bound (3.3) and Lemma 1, it suffices to show that

$$\lim_{u \to \infty} \lim_{n \to \infty} P(O \in \eta_t^{B(O,n)} \text{ for all } t \leqslant u)\, P(S \leqslant u) = 1$$

If the origin becomes vacant at some time $t \leqslant u$, then there exists a time $s \leqslant u$ such that $\eta_s^{B(O,n)} \cap \mathbb{B}_i^d = \{O\}$ for at least d indices. Since $(d^n - 1)/(d-1)$ is the number of vertices in $B(O,n) \cap \mathbb{B}^d \setminus \{O\}$,

$$P(\exists s \leqslant u \ni \eta_s^{B(O,n)} \cap \mathbb{B}^d = \{O\}) \leqslant (1 - e^{-u})^{(d^n - 1)/(d-1)}$$

It follows that $P(\exists s \leqslant u \ni \eta_s^{B(O,n)} \cap \mathbb{B}_i^d = \{O\}$ for at least d indices$) \to 0$ as n tends to infinity. Therefore,

$$\lim_{n \to \infty} P(O \in \eta_t^{B(O,n)} \text{ for all } t \leqslant u)\, P(S \leqslant u) = P(S \leqslant u)$$

Letting u tend to infinity completes the proof. □

Remark 2. The positive recurrence of A_t is in fact equivalent to finite expected extinction time of the uniform models. In order to prove this, one would construct the shape chain, a Markov chain on the finite subsets of $\mathbb{T}^d$ where isomorphic sets are identified and which has a transition from the empty set to the singleton at rate β. See Liggett[6] for the construction of the shape chain for the double parameter uniform models. The following string of equivalences proves the assertion: positive recurrence of A_t is equivalent to positive recurrence of ξ_t, which is equivalent to positive recurrence of the shape chain, which is equivalent to finite expected extinction time. The only statement which needs proof is the equivalence of positive recurrence of ξ_t and the shape chain. Given the construction of the shape chain, verifying that the reversible measure of the shape chain is summable if and only if the reversible measure of the product chain is summable proves the assertion.

4. THE FINITE EXPECTED EXTINCTION TIME THRESHOLD

By Theorem 3 and the remark following ifs proof, the positive recurrence threshold for the Markov chain A_t defined in Section 3 agrees with $\beta_1(d)$. In this section, we compute the positive recurrence threshold for A_t and thereby compute $\beta_1(d)$.

Proof of Theorem 1b. It suffices to show that A_t is positive recurrent if and only if $\beta \leqslant (1/d)((d-1)/d)^{(d-1)}$. Since A_t is reversible with respect to the measure $\pi(\cdot)$, positive recurrence is equivalent to the summability of the series

$$C(\beta) = \sum_{n=0}^{\infty} c_n \beta^n \tag{4.1}$$

where c_n is the number of $A \in \mathscr{C}^d$ such that $|A| = n$. The unique set of cardinality zero is $\{O\}$ so that $c_0 = 1$. For $n \geqslant 1$, the following recursion holds:

$$c_n = \sum_{(k_1,\ldots,k_d)} c_{k_1} \cdots c_{k_d} \tag{4.2}$$

where the sum is taken over all d-tuples in $\mathbb{N}^d$ such that $k_1 + \cdots + k_d = n-1$. To see this, note that $n \geqslant 1$ implies that x_1, the nearest neighbor of the root O, is in the set; otherwise, the set would be disconnected from O. Given that both O and x_1 are in the set, there are $n-1$ additional vertices in the set. Regarding x_1 as the root of d distinct copies of $\mathbb{B}^d$, choose $(k_1,\ldots,k_d)$ in $\mathbb{N}^d$ such that $\sum_{i=1}^d k_i = n-1$ and place $k_i + 1$ vertices (including x_1) on the ith copy of $\mathbb{B}^d$. The number of distinct arrangements of $k_i + 1$ vertices on $\mathbb{B}^d$ is c_{k_i}, which proves (4.2).

Multiplying (4.2) by β^{n-1} and taking the sum from $n = 1$ to ∞ gives

$$\frac{C(\beta) - 1}{\beta} = [C(\beta)]^d \tag{4.3}$$

Let $p(y, \beta) = \beta y^d - y + 1$. If $C(\beta) < \infty$, then $p(C(\beta), \beta) = 0$. For each $\beta > 0$, $p(\cdot, \beta)$ is a strictly convex function on $\mathbb{R}^+$ with unique minimum at $(\beta d)^{(1-d)}$. There exists a $y \in \mathbb{R}^+$ such that $p(y, \beta) = 0$ if and only if $p((\beta d)^{(1-d)}, \beta) \leqslant 0$. Furthermore, $p((\beta d)^{(1-d)}, \beta) \leqslant 0$ if and only if $\beta \leqslant (1/d)((d-1)/d)^{(d-1)}$, establishing the only if part.

Multiplying (4.2) by β^{n-1} and summing from $n = 1$ to N gives

$$\frac{C_N(\beta) - 1}{\beta} < [C_N(\beta)]^d \tag{4.4}$$

where $C_N(\beta)$ denotes the partial sum to the Nth term. Assume $\beta \leqslant (1/d)((d-1)/d)^{(d-1)}$ and let $y_1(\beta) \leqslant y_2(\beta)$ denote the two positive roots of $p(\cdot, \beta)$. By inequality (4.4), $p(C_N(\beta), \beta) > 0$. Therefore, $C_N(\beta) \in (0, y_1(\beta)) \cup (y_2(\beta), \infty)$. At $\beta = (1/d)((d-1)/d)^{(d-1)}$, $y_2(\beta) > 1$. As β decreases to 0,

$y_2(\beta)$ increases to infinity, while $C_N(\beta)$ tends to 1. Hence, the statement that $C_N(\beta) \in (y_2(\beta), \infty)$ for some $\beta \leqslant (1/d)((d-1)/d)^{(d-1)}$ contradicts the continuity of $C_N(\beta)$. Therefore, $C_N(\beta) \in (0, y_1(\beta))$ for all $\beta \leqslant (1/d) \times ((d-1)/d)^{(d-1)}$ and for all $N \in \mathbb{N}$. Let N tend to infinity to obtain $C(\beta) \leqslant y_1(\beta) < \infty$. □

It is well known in the Combinatorics literature that in case $d = 2$, the unique solution to the recursion (4.2) is the Catalan numbers, i.e.,

$$c_n = \frac{1}{n+1}\binom{2n}{n} \tag{4.5}$$

Solving (4.3) for $C(\beta)$, gives

$$C(\beta) = \frac{1 - \sqrt{1 - 4\beta}}{2\beta}$$

We choose the root with the negative sign since $\lim_{\beta \to 0} C(\beta) = 1$. Computing the power series for $C(\beta)$ centered at 0 shows that c_n is in fact the nth Catalan number. By Stirling's formula,

$$c_n \sim \frac{4^n}{\sqrt{2\pi}\, n^{3/2}}$$

which gives an alternate proof of summability up to and including 1/4 in case $d = 2$.

The technique used to compute c_n in case $d = 2$ becomes complicated and eventually breaks down. At $d = 5$, the Galois group is the entire symmetric group and therefore the roots are no longer computable by radicals However, a simple combinatorial argument can be used to compute c_n for all $d \geqslant 2$. Consider the correspondence

$$\{A \in \mathscr{C}_n^d\} \leftrightarrow \{A \in \mathscr{C}_{dn+1}^d : |\partial A| = (d-1)\, n + 1\}$$

which is given by mapping a set A of size n to the set B of size $dn + 1$ obtained by adding all vertices within distance one of A. The number of $A \in \mathscr{C}_{dn+1}^d$ with $(d-1)\, n + 1$ leaves is known to be $\binom{dn}{n}/((d-1)\, n + 1)$ (see Puha[11] for a proof). Therefore,

$$c_n = \frac{1}{(d-1)\, n + 1}\binom{dn}{n} \tag{4.6}$$

Again, an application of Stirling provides the desired summability.

5. THE COMPLETE CONVERGENCE THRESHOLD: AN EASY BOUND

If the total birth rate at a leaf is greater than the death rate, the boundary of the occupied set should have a net drift outward. Furthermore, it seems reasonable to expect this drift out at the boundary to force the occupied set to expand in all directions resulting in total occupation of the tree. We formalize this intuition and obtain an easy bound on $\beta_4(d)$.

Theorem 4. For $d \geq 2$, $\beta_4(d) \leq 1/d$.

Proof. By Theorem 3, it suffices to show that A_t is transient for $\beta > 1/d$. Modify the rates $q(A, B)$ by suppressing all births at neighbors of nonleaves. To be precise, let $\mathscr{L}^d = \{A \in \mathscr{C}^d : A \text{ has exactly one leaf}\} \cup \{O\}$ and for $A, B \in \mathscr{C}^d$ define

$$\tilde{q}(A, B) = \begin{cases} q(A, B) & \text{if } \ A, B \in \mathscr{L}^d \\ 0 & \text{otherwise} \end{cases}$$

Let L_t denote the Markov chain with state space $\mathscr{L}^d$ and rates $\{\tilde{q}(A, B)\}$. If $A_1 \in \mathscr{L}^d$, $A_2 \in \mathscr{C}^d$, $A_1 \subseteq A_2$, $x \in A_1$, and $y \notin A_2$, then

$$\tilde{q}(A_1, A_1 \backslash x) \geq q(A_2, A_2 \backslash X) \qquad \text{and} \qquad \tilde{q}(A_1, A_1 \cup y) \leq q(A_2, A_2 \cup y)$$

Therefore, we can couple L_t and A_t such that $L_t \subseteq A_t$ for all $t \geq 0$. Consequently, if L_t is transient, then so is A_t. Since $|L_t|$ is a birth and death chain with birth rate $d\beta$ and death rate one, L_t is transient for $\beta > 1/d$. $\square$

The positive recurrence and easy transience bounds of Theorem 1a and Theorem 4 are the analogs of the lower and upper bounds

$$\frac{1}{d\gamma}\left(\frac{d-1}{d(1-\gamma)}\right)^{d-1} - 1 \leq \lambda_2(d, \gamma) \leq \frac{1}{d}\left(\frac{1-\gamma d}{\gamma}\right) \tag{5.1}$$

given by Liggett[6] for the two parameter uniform model. To see this, multiply by γ and let γ decrease to 0. The technique used here to compute the positive recurrence threshold is almost the same as that used by Liggett to compute the lower bound for the double parameter uniform model. However, Liggett used a more sophisticated technique to obtain the upper bound which involved the Dirichlet principle and a notion that he called monotonicity. Essentially, he used these tools to restrict attention to the evolution of an embedded line process. Unfortunately, the simple coupling argument given here does not extend to the double parameter model.

6. THE COMPLETE CONVERGENCE THRESHOLD: IMPROVED BOUNDS

For reversible Markov chains, there is a very nice characterization of transience in terms of flows. A flow is a collection of real numbers corresponding to ordered pairs of states of the chain. Lyons[9] showed that the existence of a flow with certain properties is equivalent to transience. We exploit his result in order to prove that the chain A_t is transient for certain values of β.

In Section 6.1, a method of constructing flows with certain properties for the Markov chain A_t is described. In sections 6.2 and 6.3, particular examples of this construction are investigated. The first example leads to a nontrivial bound on $\beta_4(d)$ and proof of Theorem 1c. The second leads to a proof that $\beta_4(2) \leqslant 1/4$ and thereby a proof of Theorem 1d.

6.1. A General Strategy for Proving Transience

The purpose of this section is to outline a method for constructing flows which have special properties. We begin with the definition of a flow and the statement of the Lyons criterion. Then, we construct a class of flows which are guaranteed to satisfy all except the final condition of the Lyons criterion. Thus the problem of transience is reduced to exploring particular instances of the construction and determining for which values β the final condition holds.

Definition 1. Given a Markov chain with state space S, a *flow* on S is a collection of real numbers, or weights, $\{w(x, y)\}$ indexed by $S \times S$.

Theorem 5 (*Lyons Criterion*). Given a continuous time irreducible reversible Markov chain X_t with state space S, transition rates $q(x, y)$, and reversible measure π, transience of X_t is equivalent to the existence of a flow $\{w(x, y)\}$ on S which satisfies the following three conditions:

(i) Anti-Symmetry: For all $x, y \in S$, $w(x, y) = -w(y, x)$.

(ii) Incompressibility: There exists a $x_0 \in S$ such that

$$\sum_{y \in S} w(x_0, y) \neq 0 \qquad \text{and} \qquad \forall x \neq x_0, \qquad \sum_{y \in S} w(x, y) = 0 \tag{6.1}$$

(iii) Finite Kinetic Energy:

$$\sum_{x,\, y \in S} \frac{w^2(x, y)}{\pi(x)\, q(x, y)} < \infty \tag{6.2}$$

where, by convention, $0/0 = 0$ and $a/0 = \infty$ when $a > 0$.

Returning to the Markov chain A_t, a method for constructing flows on $\mathscr{C}^d$ which satisfy (i) and (ii) of the Lyons' criterion is outlined. Given a collection of weights $\{w(A, B)\}$, let

$$f(A) = \sum_{\{B : B \subseteq A\}} w(B, A) \qquad \text{for} \quad A \neq \{O\} \tag{6.3}$$

be the *net flow into* A (from below). Given $A \in \mathscr{C}^d$, denote the neighbors of A which contain A by $\mathscr{N}^d(A)$. For $A \in \mathscr{C}^d$, say that $r(A, \cdot)$ is a *routing vector* if the support of $r(A, \cdot)$ is contained in $\mathscr{N}^d(A)$ and $\sum_{\{B : B \in \mathscr{N}^d(A)\}} r(A, B) = 1$. Note that $r(A, \cdot)$ is not required to be nonnegative. Given a collection of routing vectors, construct the flow recursively:

(1) Set $f(\{O\}) = 1$.

(2) If $f(A)$ is defined for all $|A| < n$, for each B such that $|B| = n$ set

$$w(A, B) = f(A)\, r(A, B) \tag{6.4}$$

for all A such that $|A| \leqslant n$, where it is understood that $f(A)\, r(A, B) = 0$ when $|A| = |B| = n$. Using (6.4), $f(B)$ is now defined by (6.3) for each B such that $|B| = n$.

(3) For $A, B \in \mathscr{C}^d$ such that $|A| > |B|$, set $w(A, B) = -w(B, A)$.

Denote the collection $\{w(A, B)\}$ by F. Property (3) guarantees that F satisfies the anti-symmetry condition of the Lyons criterion. By construction, $\sum_A w(\{O\}, A) = w(\{O\}, \{O, x_1\}) = 1$. Take $B \neq \{O\}$ and combine (6.3) and (6.4) to obtain

$$\begin{aligned} \sum_{\{C \in \mathscr{C}^d : C \subseteq B\}} w(C, B) &= f(B) \\ &= f(B) \sum_{A \in \mathscr{N}^d(B)} r(B, A) \\ &= \sum_{A \in \mathscr{N}^d(B)} w(B, A) \end{aligned} \tag{6.5}$$

Since $w(B, A) = 0$ for all A such that $A \notin \mathcal{N}^d(B)$ and $B \notin \mathcal{N}^d(A)$, (6.5) proves incompressibility. This proves Proposition 1.

Proposition 1. Specifying a collection of routing vectors determines an anti-symmetric, incompressible flow.

6.2. The Uniformly Routed Flow

With this general method of constructing flows on $\mathscr{C}^d$, we attempt to a construct a flow which proves the transience of A_t for $\beta > (d-1)^{d-1}/d^d$. Using the fact that $\pi(A)\, q(A, B) = \beta^{\max(|A|, |B|)}$, the kinetic energy series is

$$\mathscr{K}(F) = 2 \sum_{n=0}^{\infty} \left(\frac{1}{\beta}\right)^{n+1} \sum_{A \in \mathscr{C}_n^d} \sum_{B \in \mathcal{N}^d(A)} w^2(A, B)$$

where $\mathscr{C}_n^d = \{A \in \mathscr{C}^d : |A| = n\}$. Since β appears in the denominator, it is natural to try to maximize the radius of convergence by minimizing the coefficients. As a first attempt, fix $A \in \mathscr{C}_n^d$ and

$$\text{minimize} \quad \sum_{B \in \mathcal{N}^d(A)} w^2(A, B) \quad \text{subject to} \quad f(A) = \sum_{B \in \mathcal{N}^d(A)} w(A, B)$$

The solution to this minimization problem is to set

$$w(A, B) = \frac{f(A)}{|\mathcal{N}^d(A)|}$$

Since $|\mathcal{N}^d(A)| = (d-1)\,|A| + 1$,

$$r(A, B) = \frac{1}{(d-1)\,|A| + 1} \qquad \forall B \in \mathcal{N}^d(A) \tag{6.6}$$

In this case, the routing vectors are nonnegative. Let $h(n) = \sum_{A \in \mathscr{C}_n^d} f^2(A)$. If $r(A, B)$ is defined by (6.6), then

$$\mathscr{K}(F) = 2 \sum_{n=0}^{\infty} \frac{h(n)}{\beta^{n+1}((d-1)\,n+1)}$$

By Theorems 5 and 3,

$$\beta_4(d) \leqslant \limsup_{n \to \infty} h(n)^{1/n} \tag{6.7}$$

Theorem 1c will be a consequence of obtaining bounds on the limiting behavior of the sequence $h(n)^{1/n}$.

The first thing to note is that f can be computed exactly (see the next lemma). However, we will not be able to compute h explicitly. Instead, using the expression for f, h is expressed as a ratio. The goal is to prove that the sequence $h(n)$ is bounded above and below by sequences for which the associated power series have the same radius of convergence. Therefore, determining the radius of convergence of $\mathscr{K}(F)$ will be equivalent to determining the radius of convergence for a power series with coefficients equal to either the upper or lower bound. The bounds are chosen so that the numerators agree with the numerators of $h(n)$. The reason for choosing the bounds this way is to exploit the fact that the numerators of $h(n)$ satisfy a nice recursion. By choosing the denominator of the lower bound appropriately, the numerator recursion will guarantee that the lower bound satisfies a related recursion. The fact that the lower bound satisfies this related recursion allows one to obtain bounds on the radius of convergence of the power series with coefficients which agree with the lower bound.

We begin by finding an explicit expression for f. Then a combinatorial lemma is stated. As a consequence of this lemma, the numerator recursion for the sequence $h(n)$ is obtained. Next, the sequences which bound $h(n)$ are introduced. Finally, bounds on the radius of convergence of the power series with coefficients which agree with the lower bound are obtained for $d \geqslant 2$. This bound is an improvement over the easy transience bound of $1/d$ if and only if $d \geqslant 4$.

Definition 2. An *increasing path* from $\{O\}$ to A in $\mathscr{C}^d$ is a collection $\{B_i\}_{i=0}^{|A|}$ of sets in $\mathscr{C}^d$ such that $B_0 = \{O\}$, $B_{|A|} = A$, and $B_{i+1} \in \mathscr{N}^d(B_i)$ for $i = 0, \ldots, |A| - 1$. Let $N(A)$ be the number of paths which increase from $\{O\}$ to A.

Lemma 2. If $r(A, B)$ is defined by (6.6), then for A such that $|A| = n \geqslant 1$,

$$f(A) = \frac{N(A)}{\prod_{k=0}^{n-1} ((d-1)k+1)}$$

Proof. If $|A| = 1$, then $A = \{O, x_1\}$. Since $r(\{O\}, \{O, x_1\}) = 1$, (6.3) gives $f(\{O, x_1\}) = 1$ as desired. Assume that the assertion holds for $|A| < n$. If $|A| = n$, then, by (6.3) and (6.4),

$$\begin{aligned}
f(A) &= \sum_{\{B : B \subseteq A\}} w(B, A) \\
&= \sum_{\{B : B \subseteq A\}} f(B)\, r(B, A) \\
&= \sum_{\{B : A \in \mathcal{N}^d(B)\}} \frac{N(B)}{\prod_{k=0}^{n-2} ((d-1)\, k+1)} \frac{1}{(d-1)(n-1)+1} \\
&= \frac{1}{\prod_{k=0}^{n-1} ((d-1)\, k+1)} \sum_{\{B : A \in \mathcal{N}^d(B)\}} N(B) \\
&= \frac{N(A)}{\prod_{k=0}^{n-1} ((d-1)\, k+1)} \qquad \square
\end{aligned}$$

Lemma 3. For $n \geqslant 1$, there exists a one-to-one correspondence between $\mathscr{C}_n^d$ and the disjoint union $\bigsqcup_{(j_1,\dots,j_d)} \mathscr{C}_{j_1}^d \times \cdots \times \mathscr{C}_{j_d}^d$, where the union runs over all d-tuples in $\mathbb{N}^d$ with $j_1 + \cdots + j_d = n-1$, such that under this correspondence

$$N(A) = \binom{|A|-1}{|A_1|,\dots,|A_d|} N(A_1) \cdots N(A_d) \tag{6.8}$$

Proof. Let $\{y_1,\dots,y_d\}$ denote the nearest neighbors of x_1 in $\mathbb{B}^d$. Set $\mathbb{B}_{1i}^d = \{y : \|y_i - y\| \leqslant \|x_1 - y\|\} \cup x_1$ and $A_i = A \cap \mathbb{B}_{1i}^d$. Since $\mathbb{B}_{1i}^d \cong \mathbb{B}^d$ for $1 \leqslant i \leqslant d$, $A \leftrightarrow (A_1,\dots,A_d)$. Equation (6.9) is an immediate consequence of this correspondence. $\square$

Squaring (6.9),

$$N^2(A) = \binom{|A|-1}{|A_1|,\dots,|A_d|}^2 N^2(A_1) \cdots N^2(A_d) \tag{6.9}$$

for $A \in \mathscr{C}^d$ such that $|A| \geqslant 1$. For $n \in \mathbb{N}$, let $N_n = \sum_{A \in \mathscr{C}_n^d} N^2(A)$. Summing (6.9) over all ordered d-tuples $(A_1,\dots,A_d) \in \mathscr{C}^d \times \cdots \times \mathscr{C}^d$ such that $|A_1| + \cdots + |A_d| = n-1$ gives

$$N_n = \sum_{(j_1,\dots,j_d)} \binom{n-1}{j_1,\dots,j_d}^2 N_{j_1} \cdots N_{j_d} \qquad \text{for} \quad n \geqslant 1 \tag{6.10}$$

where the sums runs over all d-tuples in $\mathbb{N}^d$ with $j_1 + \cdots + j_d = n-1$. By definition of $h(n)$, Lemma 2, and definition of N_n,

$$h(n) = \frac{N_n}{\prod_{k=0}^{n-1} ((d-1)\, k+1)^2}$$

If we could solve recursion (6.10), then we would be able to compute $h(n)$ exactly. We pursue an alternate strategy and use (6.10) to obtain information about the asymptotic behavior of $h(n)$. For $n \geqslant 1$,

$$(d-1)^{n-1}(n-1)! \leqslant \prod_{k=0}^{n-1} ((d-1)k+1) \leqslant (d-1)^n n!$$

Set $\ell(0)=1$ and $u(0)=1$. For $n \geqslant 1$, set

$$\ell(n) = \frac{N_n}{(d-1)^{2n} n!^2} \qquad \text{and} \qquad u(n) = \frac{N_n}{(d-1)^{2(n-1)}(n-1)!^2}$$

Then

$$\ell(n) \leqslant h(n) \leqslant u(n) \qquad \text{for all} \quad n \in \mathbb{N} \tag{6.11}$$

Furthermore, for $n \geqslant 1$, $u(n) = (d-1)^2 n^2 \ell(n)$ so that

$$\limsup_{n \to \infty} \ell(n)^{1/n} = \limsup_{n \to \infty} u(n)^{1/n} \tag{6.12}$$

Combining (6.11) and (6.12) proves the next proposition.

Proposition 2. $\limsup_{n \to \infty} h(n)^{1/n} = \limsup_{n \to \infty} \ell(n)^{1/n}$.

Proposition 3. For $n \geqslant 1$,

$$\ell(n) = \frac{1}{(d-1)^2 n^2} \sum_{(j_1, \ldots, j_d)} \ell(j_1) \cdots \ell(j_d) \tag{6.13}$$

where the sum runs over all d-tuples in $\mathbb{N}^d$ with $j_1 + \cdots + j_d = n-1$.

Proof. Divide (6.10) by $(d-1)^{2n} n!^2$. □

Finally, solving (6.13) is equivalent to solving a modified recursion. Suppose that $\tilde{\ell}(0)=1$ and for $n \geqslant 1$, $\tilde{\ell}(n)$ satisfies

$$\tilde{\ell}(n) = \frac{1}{n^2} \sum_{(j_1, \ldots, j_d)} \tilde{\ell}(j_1) \cdots \tilde{\ell}(j_d) \tag{6.14}$$

where the sum runs over all d-tuples in $\mathbb{N}^d$ with $j_1 + \cdots + j_d = n-1$. Then $\ell(n) = \tilde{\ell}(n)/(d-1)^{2n}$ satisfies (6.13). Therefore, obtaining bounds on the solution of (6.14) gives bounds on the solution of (6.13).

Theorem 6. For $n \geqslant 1$,

$$\tilde{\ell}(n) \leqslant \frac{1}{n}\left(\frac{d}{n}\right)^{n-1}$$

Proof of Theorem 1c. By Theorem 6, the relationship between solutions of (6.13) and (6.14), Proposition 2, and inequality (6.7), $\beta_4(d) \leqslant d/(2(d-1)^2)$. □

Theorem 6 is proved by induction. In order to execute the induction step, the following lemma is needed. This lemma is a special case of a well known expansion of the binomial coefficient $\binom{k+n-1}{n}$ with $k=2$.

Lemma 4. For any positive integer n,

$$\sum_{j=1}^{n} \frac{2^j}{j!} \sum_{\gamma \in \Gamma(n,j)} \frac{1}{\gamma_1 \cdots \gamma_j} = n+1 \tag{6.15}$$

where $\Gamma(n, j)$ is the set of all ordered partitions of n of into j parts and γ_i is the ith element in the partition γ.

Proof. Writing $-\log(1-x)$ as a power series centered at 0 gives

$$(-\log(1-x))^j = \left(\sum_{k=1}^{\infty} \frac{x^k}{k}\right)^j = \sum_{n=j}^{\infty} \sum_{\gamma \in \Gamma(n,j)} \frac{1}{\gamma_1 \cdots \gamma_j} x^n \qquad \text{for} \quad |x|<1$$

Let $k \in \mathbb{N}$. For $|x|<1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \binom{k+n-1}{n} x^n &= \frac{1}{(1-x)^k} - 1 \\
&= e^{-k\log(1-x)} - 1 \\
&= \sum_{j=1}^{\infty} \frac{(-k\log(1-x))^j}{j!} \\
&= \sum_{j=1}^{\infty} \frac{k^j}{j!} \sum_{n=j}^{\infty} \sum_{\gamma \in \Gamma(n,j)} \frac{1}{\gamma_1 \cdots \gamma_j} x^n \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{k^j}{j!} \sum_{\gamma \in \Gamma(n,j)} \frac{1}{\gamma_1 \cdots \gamma_j} x^n
\end{aligned}$$

Taking $k=2$ completes the proof. □

Proof of Theorem 6. By (6.14), $\tilde{\ell}(1)=1$ which verifies the assertion for $n=1$. Assume that the assertion holds for $m<n$. We have

$$\begin{aligned}\tilde{\ell}(n) &= \frac{1}{n^2} \sum_{(m_1,\ldots,m_d)} \tilde{\ell}(m_1)\cdots\tilde{\ell}(m_d) \\ &= \frac{1}{n^2} \sum_{j=1}^{\min(d,n-1)} \binom{d}{j} \sum_{\gamma\in\Gamma(n-1,j)} \tilde{\ell}(\gamma_1)\cdots\tilde{\ell}(\gamma_j)\end{aligned}$$

since $\tilde{\ell}(m_i)=1$ when $m_i=0$. By assumption,

$$\begin{aligned}\tilde{\ell}(n) &\leqslant \frac{1}{n^2} \sum_{j=1}^{\min(d,n-1)} \binom{d}{j} \sum_{\gamma\in\Gamma(n-1,j)} \left(\frac{d}{2}\right)^{n-1-j} \frac{1}{\gamma_1\cdots\gamma_j} \\ &\leqslant \frac{1}{n^2} \sum_{j=1}^{\min(d,n-1)} \frac{d^j}{j!}\left(\frac{d}{2}\right)^{n-1-j} \sum_{\gamma\in\Gamma(n-1,j)} \frac{1}{\gamma_1\cdots\gamma_j} \\ &= \frac{1}{n^2}\left(\frac{d}{2}\right)^{n-1} \sum_{j=1}^{\min(d,n-1)} \frac{2^j}{j!} \sum_{\gamma\in\Gamma(n-1,j)} \frac{1}{\gamma_1\cdots\gamma_j} \\ &\leqslant \frac{1}{n^2}\left(\frac{d}{2}\right)^{n-1} \sum_{j=1}^{n-1} \frac{2^j}{j!} \sum_{\gamma\in\Gamma(n-1,j)} \frac{1}{\gamma_1\cdots\gamma_j}\end{aligned}$$

By Lemma 4,

$$\tilde{\ell}(n) \leqslant \frac{1}{n}\left(\frac{d}{2}\right)^{n-1} \qquad \square$$

A simple computation provides evidence that for large d the bound given in Theorem 1c is close to the best that this flow achieves. Thus, not so much is lost in the inequality in Theorem 6. Let A_n be the Markov chain on $\mathscr{C}^d$ with transition probabilities defined by (6.6) and let ℓ_n be the number of leaves in the set A_n. By conditioning on ℓ_{n-1}, one gets a recursion which leads to

$$\mathbb{E}(\ell_n) = \frac{(d-1)\,n+1}{2d-1} \qquad n\geqslant 2$$

In other words, the typical set that the uniform flow visits has a death rate which is roughly the birth rate divided by $(2d-1)\,\beta$. In $d=2$, these sets are not only typical, but rather uniform flow visits them with very high probability:

$$\mathbb{E}(\ell_n^2) = \frac{(n+1)(5n+7)}{45} \qquad n\geqslant 4, \quad d=2$$

and therefore,

$$\frac{\ell_n}{(d-1)\,n+1} \xrightarrow{P} \frac{1}{2d-1} \qquad \text{for} \quad d=2$$

For small d the bound given in Theorem 1c is much worse than $1/(2d-1)$. However, by handling the cases $d=2$ and $d=3$ separately, the bound induced on $\ell(n)$ by Theorem 6 can be improved to $14(n+1)(1/3)^{n+2}$ and $(1/5)^{n-1}$ respectively. We conjecture that $1/(2d-1)$ is the optimal bound for this flow. Numerical evidence suggests that one cannot hope for much better.

6.3. The Uniformly Distributed Flow

In the previous section, the main goal became to determine the asymptotic behavior of $h(n)$. This resulted from the fact that $\sum_{B \in \mathcal{N}^d(A)} r^2(A, B) = 1/((d-1)\,n+1)$, and therefore, the presence of this factor did not affect the radius of convergence of $\mathcal{K}(F)$. If we require the routing vectors to be absolutely bounded by b, then

$$\frac{1}{(d-1)\,n+1} \leqslant \sum_{B \in \mathcal{N}^d(A)} r^2(A, B) \leqslant b^2((d-1)\,n+1) \tag{6.16}$$

Thus, under the assumption that routing vector are bounded, the asymptotic behavior of $h(n)$ governs the radius of convergence of $\mathcal{K}(F)$.

As a consequence of the construction, $\sum_{A \in \mathcal{C}_n^d} f(A) = 1$. Hence, we seek to minimize a quadratic function subject to a linear constraint. If this linear constraint were the only constraint, then the solution would be to partition 1 into equal parts, i.e., distribute the fluid uniformly over sets of size n. However, we require the the flow to be incompressible which introduces many additional constraints. Notice that if a flow exists with bounded routing vectors such that $f(A) = 1/c_{|A|}$, then by (6.16),

$$\mathcal{K}(F) \leqslant \frac{2b^2}{\beta} \sum_{n=0}^{\infty} \frac{(d-1)\,n+1}{c_n \beta^n} \tag{6.17}$$

This series is summable for $\beta > (1/d)((d-1)/d)^{(d-1)}$ since, except for the factor of $(d-1)\,n+1$, each term is the exact reciprocal of the terms appearing in (4.1). Due the these observations, we attempt to construct a uniformly distributed flow with bounded routing vectors.

Suppose that one has constructed routing vectors bounded by b such that $f(A) = 1/c_{|A|}$ for all $A \in \mathcal{C}^d$ such that $|A| < n$. Exploit the fact that $\mathcal{C}_n^d$

is in one-to-one correspondence with $\bigsqcup_{(k_1,\ldots,k_d)} \mathscr{C}^d_{k_1} \times \cdots \times \mathscr{C}^d_{k_d}$ where the union runs over all d-tuples in $\mathbb{N}^d$ such that $k_1 + \cdots + k_d = n-1$ and use the routing vectors $\{r(A, \cdot)\}_{|A|<n}$ to construct the routing vectors for $\mathscr{C}^d_n$. More specifically, associate to each set a *preliminary routing vector* $\alpha(A, i)$ which determines the amount of fluid routed to branch i in set A. In particular, let $\alpha(A, \cdot)$ be such that $\sum_{i=1}^d \alpha(A, i) = 1$. If A corresponds to $(A_1,\ldots, A_d)$, B corresponds to $(B_1,\ldots, B_d)$, $B \in \mathscr{N}^d(A)$, and $A_i \neq B_i$, then let

$$r(A, B) = \alpha(A, i)\, r(A_i, B_i)$$

Since

$$\begin{aligned}\sum_{\{B \in \mathscr{N}^d(A)\}} r(A, B) &= \sum_{i=1}^{d} \sum_{\{B_i \in \mathscr{N}^d(A_i)\}} \alpha(A_i)\, r(A_i, B_i)\\ &= \sum_{i=1}^{d} \alpha(A, i) \sum_{\{B_i \in \mathscr{N}^d(A_i)\}} r(A_i, B_i)\\ &= \sum_{i=1}^{d} \alpha(A, i) = 1 \qquad (6.18)\end{aligned}$$

it follows that $r(A, \cdot)$ is a routing vector. Furthermore, if $|\alpha(A, i)| \leqslant 1$, then $r(A, B)$ is bounded by b. Therefore, in order to specify a collection of bounded routing vectors, it suffices to specify a collection $\alpha(A, i)$ of preliminary routing vectors which are absolutely bounded by one.

A priori, one might expect $\alpha(A, \cdot)$ to depend on the entire structure of A. However, it is reasonable to expect dependence only on the cardinalities of A_j for $1 \leqslant j \leqslant d$. One explanation for this is that the distribution which we are trying to achieve depends only on cardinality. A more practical reason for making this assumption is that it simplifies the set of equations that $\alpha(A, \cdot)$ must satisfy by allowing a second application of the induction hypothesis. For $k \in \mathbb{N}^d$ such that $k_1 + \cdots + k_d = n-1$, let $\alpha_i(n; k)$ be a preliminary routing vector in a set A when $|A| = n$ and $|A_j| = k_j$ for $1 \leqslant j \leqslant d$. Thus, the function $\alpha_i(n; k)$ must satisfy

$$\alpha_1(n; k) + \cdots + \alpha_d(n; k) = 1 \qquad \text{and} \qquad |\alpha_i(n; k)| \leqslant 1 \qquad (6.19)$$

for all $n \geqslant 1$ and $k \in \mathbb{N}^d$ such that $k_1 + \cdots + k_d = n-1$. Also, require that for all permutations σ of d objects $\alpha_{\sigma(i)}(n; \sigma(k)) = \alpha_i(n; k)$ where σ acts on a d-vector in the usual manner by permuting the indices. This condition simply states that the preliminary routing vectors are invariant under automorphisms of $\mathbb{B}^d$. For all $A \in \mathscr{C}^d_n$ and $B \in \mathscr{N}^d(A)$, set

$$r(A, B) = \alpha_i(n; k)\, r(A_i, B_i) \qquad \text{if} \quad A_i \neq B_i \qquad (6.20)$$

where $|A| = n$ and $|A_j| = k_j$ for all $1 \leqslant j \leqslant d$. The goal is to choose $\alpha_i(n; \cdot)$ such that the flow is distributed uniformly over sets of size $n+1$.

For $B \in \mathscr{C}^d_{n+1}$, set $k_i = |B_i|$. Make the convention that $c_{-1} = 0$. The net flow into B is given by

$$\begin{aligned} f(B) &= \sum_{\{A : B \in \mathscr{N}^d(A)\}} f(A)\, r(A, B) \\ &= \frac{1}{c_n} \sum_{i=1}^{d} \sum_{\{A_i : B_i \in \mathscr{N}^d(A_i)\}} \alpha_i(n; k - e_i)\, r(A_i, B_i) \\ &= \frac{1}{c_n} \sum_{i=1}^{d} \alpha_i(n; k - e_i)\, c_{k_i - 1} \sum_{\{A_i : B_i \in \mathscr{N}^d(A_i)\}} f(A_i)\, r(A_i, B_i) \\ &= \frac{1}{c_n} \sum_{i=1}^{d} \alpha_i(n; k - e_i) \frac{c_{k_i - 1}}{c_{k_i}} \end{aligned} \tag{6.21}$$

where e_i is the d-vector with all entries equal 0 except the ith which is 1.

Lemma 5. If, for each $n \geqslant 1$, there exists $\alpha_i(n; \cdot)$ satisfying (6.19) and

$$\frac{c_n}{c_{n+1}} = \sum_{i=1}^{d} \alpha_i(n; k - e_i) \frac{c_{k_i - 1}}{c_{k_i}} \tag{6.22}$$

for all $k \in \mathbb{N}^d$ such that $k_1 + \cdots + k_d = n$, then $\beta_1(d) = \beta_4(d)$.

Proof. Set $r(\{O\}, \{O, x_1\}) = 1$. For $|A| \geqslant 1$, define $r(A, \cdot)$ recursively by (6.20). By induction, $|r(A, \cdot)| \leqslant 1$. By (6.18), $r(A, \cdot)$ makes up a collection of routing vectors. By (6.21) and (6.22), $f(A) = 1/c_{|A|}$ for all $A \in \mathscr{C}^d$. Therefore, (6.17) implies finite kinetic energy for $\beta > (1/d)((d-1)/d)^{d-1}$. By Theorems 3 and 5, $\beta_4(d) \leqslant (1/d)((d-1)/d)^{d-1}$. Combining this with Theorem 1b and the fact that $\beta_1(d) \leqslant \beta_4(d)$ completes the proof. □

Restrict attention to the case $d = 2$. Set $\rho(j) = c_j / c_{j+1}$. By the assumption that $\alpha_i(n; \cdot)$ is invariant under automorphisms of $\mathbb{B}^d$, it suffices to define $\alpha_1(n; k)$ for all $n \geqslant 1$ and for all $k \in \mathbb{N}^2$ such that $k_1 + k_2 = n - 1$. If, for all $n \geqslant 1$, $\alpha_i(n; \cdot)$ is a solution of

$$\begin{aligned} &1 = \alpha_1(n; (j, n-1-j)) + \alpha_2(n; (j, n-1-j)) \qquad |\alpha_i(n; (j, n-1-j))| \leqslant 1 \\ &\rho(n) = \alpha_1(n; (n-1, 0))\, \rho(n-1) \\ &\rho(n) = \alpha_1(n; (j-1, n-j))\, \rho(j-1) + \alpha_2(n; (j, n-j-1))\, \rho(n-j-1) \end{aligned} \tag{6.23}$$

where $0 \leqslant j \leqslant n-1$, then Lemma 5 implies that $\beta_1(2) = \beta_4(2)$. By substituting $1 - \alpha_1(n; (j, n-1-j))$ for $\alpha_2(n; (j, n-j-1))$ in the final equation, solving (6.23) equivalent to solving

$$\begin{aligned} &\alpha_1(n; (j, n-1-j)) \geqslant 0 \qquad \text{for} \quad 0 \leqslant j \leqslant n-1 \\ &\alpha_1(n; (n-1, 0)) = \frac{\rho(n)}{\rho(n-1)} \\ &\alpha_1(n; (j-1, n-j)) = \frac{\rho(n) - \rho(n-j-1) + \alpha_1(n; (j, n-1-j))\, \rho(n-1-j)}{\rho(j-1)} \end{aligned} \tag{6.24}$$

for $1 \leqslant j \leqslant n-1$ and $n \geqslant 1$.

Theorem 7. The unique solution of (6.24) is

$$\alpha_1(n; (j, n-1-j)) = \frac{(j+1)(2j+1)(3n-2j)}{n(n+1)(2n+1)} \tag{6.25}$$

In particular, $\beta_1(2) = \beta_4(2)$.

Proof. Using the fact that $c_{j+1} = (4j+2)\, c_j/(j+2)$, it follows that $\rho(j) = (j+2)/(4j+2)$ and therefore

$$\frac{\rho(n)}{\rho(n-1)} = \frac{(2n-1)(n+2)}{(n+1)(2n+1)}$$

Take $j = n-1$ in the righthand side of (6.25) to verify the base case. Assume that (6.25) holds for all m such that $j \leqslant m \leqslant n-1$. Then

$$\begin{aligned} &\alpha_1(n; (j-1, n-j)) \\ &\quad = \frac{\rho(n) - \rho(n-j-1) + \alpha_1(n; (j, n-j-1))\, \rho(n-1-j)}{\rho(j-1)} \\ &\quad = \frac{4j-2}{j+1} \left(\frac{3(j+1)}{-8n^2 + 8jn + 4j + 2} + \frac{(j+1)(2j+1)(3n-2j)(n-j+1)}{n(n+1)(2n+1)(4n-4j-2)} \right) \\ &\quad = \frac{4j-2}{j+1} \frac{j(j+1)(3n-2(j-1))}{4n^3 + 6n + 2n} \\ &\quad = \frac{(2j-1)\, j(3n-2(j-1))}{2(2n+1)(n+1)} \end{aligned}$$

which proves the result. □

Lemma 5 reduces proving Conjecture 1 to proving that a solution to (6.19) and (6.22) exists for all $d \geqslant 3$. The main obstacle in proving a solution exists for $d \geqslant 3$ is that, disregarding the absolute bound of one requirement, the solution to (6.19) and (6.22) is not unique (see Puha[11]). Therefore, verifying that a suitably bounded solution exists for all $n \in \mathbb{N}$ becomes more challenging. The next section is devoted to providing heuristic support for the existence of a solution to (6.19) and (6.22) for all $d \geqslant 3$.

7. THE LIMITING VERSION

Equations (6.19) and (6.22) make up a collection of linear algebra problems indexed by $\mathbb{N}$. Each problem has a distinct set of variables. Therefore, a solution to the $n = 5$ problem need not relate to a solution of the $n = 6$ problem. However, given the similarity of the equations it seems reasonable to conclude that there exists a collection of solutions which are consistent in some sense. Any reasonable consistency condition will imply that the limit as n tends to infinity of $\alpha_i(n; \cdot)$ exists.

We investigate the limiting version of the equations (6.19) and (6.22). Under the limiting operation, Eq. (6.22) becomes a first order partial differential equation. It turns out that for all $d \geqslant 2$, the limiting version of (6.19) and (6.22) has a solution which is absolutely bounded by one. The existence of such a solution provides evidence that solutions to (6.19) and (6.22) exist which are absolutely bounded by one. Proving that such solutions exists, in turn proves Conjecture 1.

Here, a study of the limiting version of (6.19) and (6.22) is presented as support for the conjecture. In Section 7.1, the continuous problem is derived. In Section 7.2, the method used to find the solution is explained. The main idea is to assume that the solution can be expressed as a series and to devise a method for computing the coefficients. As one might expect, this approach becomes excessively complicated in general. However, the approach does provide an answer for small d and an educated guess for the general problem. An independent proof of Theorem 2 presented in Section 7.3.

7.1. The Derivation of the Continuous Problem

Assume that $\{\alpha(n, \cdot)\}_{n \in \mathbb{N}}$ is a set of solutions to the discrete problem such that

$$\alpha^*(x_1, ..., x_d) = \lim_{n \to \infty} \alpha_n^*(x_1, ..., x_d) \tag{7.1}$$

exists where $\alpha_n^*(x_1,..., x_d) = \alpha_1(\lfloor nx_1 \rfloor + \cdots + \lfloor nx_d \rfloor + 1; \lfloor nx_1 \rfloor,..., \lfloor nx_d \rfloor)$. By definition, $\alpha^*(x_1,..., x_d)$ is symmetric in the variables $(x_2,..., x_d)$. Furthermore, $\alpha^*(x_1,..., x_d) = \alpha^*(ax_1,..., ax_d)$ for all $a > 0$. Therefore,

$$\alpha^*(x_1,..., x_d) = v\left(\frac{x_2}{x_1 + \cdots + x_d},..., \frac{x_d}{x_1 + \cdots + x_d}\right) \tag{7.2}$$

for some symmetric function v defined on the $d-1$ dimensional simplex $\mathbb{S}^{d-1}$. The limit of (31) is given by

$$\sum_{i=1}^{d} \alpha^*(x_i, x_1,..., x_{i-1}, x_{i+1},..., x_d) = 1 \tag{7.3}$$

Letting $s_i = x_i/(x_1 + \cdots + x_d)$ for $1 \leqslant i \leqslant d$ and expressing (7.3) in terms of v,

$$v(s_2,..., s_d) + \cdots + v(s_1,..., s_{i-1}, s_{i+1},..., s_d) + \cdots + v(s_1,..., s_{d-1}) = 1 \tag{7.4}$$

If one simply takes the limit of Eq. (6.22), it collapses into (7.3). Therefore, first order information must be considered. By computing the first two coefficients of the power series centered at infinity,

$$\frac{c(n)}{c(n+1)} \sim \frac{1}{d}\left(\frac{d-1}{d}\right)^{d-1}\left(1 + \frac{3}{2n}\right)$$

Expressing (6.22) in terms of α_n^* gives,

$$\sum_{i=1}^{d} \alpha_n^*\left(x_i - \frac{1}{n}, x_1,..., x_{i-1}, x_{i+1},..., x_d\right) \frac{c(\lfloor n(x_i - (1/n))\rfloor)}{c(\lfloor nx_i \rfloor)} = \frac{c(\lfloor nx_1 \rfloor + \cdots + \lfloor nx_d \rfloor)}{c(\lfloor nx_1 \rfloor + \cdots + \lfloor nx_d \rfloor + 1)} \tag{7.5}$$

Asymptotically, (7.5) is given by

$$\sum_{i=1}^{d} \alpha_n^*\left(x_i - \frac{1}{n}, x_1,..., x_{i-1}, x_{i+1},..., x_d\right)\left(1 + \frac{3}{2n(x_i - (1/n))}\right) = 1 + \frac{3}{2n(x_1 + \cdots + x_d)} + o\left(\frac{1}{n}\right) \tag{7.6}$$

As previously mentioned, first order information must be retained. Therefore, (7.6) will be multiplied by n. In order to prevent both sides from tending to infinity, (6.19) is subtracted from (7.6) before multiplication by n. This gives

$$\sum_{i=1}^{d} n\left(\alpha_n^*\left(x_i-\frac{1}{n}, x_1,..., x_{i-1}, x_{i+1},..., x_d\right)-\alpha_n^*(x_i, x_1,..., x_{i-1}, x_{i+1},..., x_d)\right)$$
$$+\frac{3}{2}\sum_{i=1}^{d}\frac{\alpha_n^*(x_i-(1/n), x_1,..., x_{i-1}, x_{i+1},..., x_d)}{x_i-(1/n)}=\frac{3}{2(x_1+\cdots+x_d)}+o(1)$$

Therefore, in the limit, (6.22) becomes

$$\sum_{i=1}^{d}\left(\frac{3}{2x_i}-\frac{\partial}{\partial x_i}\right)\alpha^*(x_i, x_1,..., x_{i-1}, x_{i+1},..., x_d)=\frac{3}{2(x_1+\cdots+x_d)} \tag{7.7}$$

On the $d-1$ dimensional simplex, let

$$T_d v(s_1,..., s_{d-1})=\frac{v(s_1,..., s_{d-1})}{1-s_1-\cdots-s_{d-1}}+\frac{2}{3}\sum_{i=1}^{d-1} s_i\frac{\partial}{\partial s_i}v(s_1,..., s_{d-1})$$

Multiplying (7.7) by $2(x_1+\cdots+x_d)/3$ and expressing (7.7) in terms of $T_d v$,

$$\sum_{i=1}^{d} T_d v(s_1,..., s_{i-1}, s_{i+1},..., s_d)=1 \tag{7.8}$$

For w: $\mathbb{S}^{d-1}\to\mathbb{R}$, let $L_d w$: $\partial\mathbb{S}^d\to\mathbb{R}$ be defined by

$$L_d w(s_1,..., s_d)=\sum_{i=1}^{d} w(s_1,..., s_{i-1}, s_{i+1},..., s_d) \tag{7.9}$$

Equations (7.4) and (7.8) can be expressed in terms of L_d as

$$L_d v(s_1,..., s_d)=1 \qquad \text{and} \qquad L_d T_d v(s_1,..., s_d)=1 \tag{7.10}$$

respectively. The next proposition which summarizes the statement of the continuous problem has been proved.

Proposition 4. If α^*: $\mathbb{R}_+^d\to\mathbb{R}$ is symmetric in the variables $x_2,..., x_d$ and satisfies (7.3) and (7.7), then v: $\mathbb{S}^{d-1}\to\mathbb{R}$ defined by (7.2) is a symmetric solution of (7.10). Conversely, if v: $\mathbb{S}^{d-1}\to\mathbb{R}$ is symmetric and satisfies (7.10), then α^*: $\mathbb{R}_+^d\to\mathbb{R}$ defined by (7.2) is symmetric in the variables $x_2,..., x_d$ and satisfies (7.3) and (7.7).

7.2. The Method for Finding a Solution

The method used to actually find the solution is presented in this section. The strategy is to express a candidate solution as a series with unknown coefficients and to use the PDE to determine the coefficients. The approach is demonstrated in $d=3$ and only the main ideas are presented here. For a more detailed account, see Puha.[11] In section 7.3, a complete proof of Theorem 2 is given which is independent of the approach taken here.

The goal is to find $v(s, t)$ such that

$$v(s, t)+v(1-s-t, t)+v(1-s-t, s)=1$$
$$T_3 v(s, t)+T_3 v(1-s-t, t)+T_3 v(1-s-t, s)=1$$

For any such $v(s, t)$, $v(s, t)=u(s, t)+1/3$ for some $u(s, t)$ which satisfies

$$u(s, t)+u(1-s-t, t)+u(1-s-t, s)=0 \tag{7.11}$$

A collection of symmetric polynomials which satisfy (7.11) is given by

$$\begin{aligned} u_{n, m}(s, t) &= (1-s-t)^n (st)^n ((1-t-2s) t^m + (1-s-2t) s^m) \\ &= (1-s-t)^{n+1} (st)^n (s^m+t^m) \\ &\quad - (1-s-t)^n (st)^{n+1} (s^{m-1}+t^{m-1}) \end{aligned}$$

where $m, n \in \mathbb{N}$. Consider $u(s, t)=\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \omega_{n, m} u_{n, m}(s, t)$ where $\omega_{n, m} \in \mathbb{R}$. The goal is to choose $\omega_{n, m}$ such that $T_3(u+1/3)(s, t)-1/3$ satisfies (7.11).

Since $T_3 u_{n, m}(s, t)$ is not expressible in terms of the collection $\{u_{n, m}(s, t)\}_{n, m \in \mathbb{N}}$, some symmetric polynomials are added to the collection. Let

$$p_{n, m}(s, t)=(1-s-t)^n (st)^n (s^m+t^m)$$

The collection $\{u_{n, m}(s, t), p_{n, m}(s, t)\}_{n, m \in \mathbb{N}}$ spans the set of all symmetric polynomials in two variables. It turns out that

$$\begin{aligned} &T_3 u_{n, m}(s, t) \\ &\quad = \frac{2(3n+m+1)}{3} u_{n, m}(s, t) - \left(\frac{3-2n}{6}\right) (u_{n-1, m+2}(s, t) - u_{n-1, m+1}(s, t)) \\ &\quad + \left(\frac{4-4n}{3}\right) p_{n, m}(s, t) - \left(\frac{3-2n}{6}\right) (p_{n-1, m+1}(s, t) \\ &\quad - 2p_{n-1, m+2}(s, t) + p_{n-1, m+3}(s, t)) \end{aligned}$$

Also,

$$T_3(1/3)(s,t)-1/3=\frac{s+t}{3(1-s-t)}=\frac{p_{-1,2}(s,t)-p_{-1,3}(s,t)-u_{-1,2}(s,t)}{6}$$

Recall that the objective is to choose $\omega_{n,m}$ such that $T_3(u+1/3)(s,t)-1/3$ satisfies (7.11). In other words, the coefficient of $p_{n,m}(s,t)$ in $T_3(u+1/3)(s,t)-1/3$ should be zero. If $\kappa_{m,n}$ denotes the coefficient of $p_{n,m}(s,t)$ in $T_3(u+1/3)-1/3$, it follows that

$$\kappa(n,m)=\begin{cases}\frac{1}{6}+\frac{8\omega(-1,2)}{3}-\frac{\omega(0,1)}{2}+\omega(0,0)-\frac{\omega(0,-1)}{2} \\ \quad\text{if } (n,m)=(-1,2) \\ -\frac{1}{6}+\frac{8\omega(-1,3)}{3}-\frac{\omega(0,2)}{2}+\omega(0,1)-\frac{\omega(0,0)}{2} \\ \quad\text{if } (n,m)=(-1,3) \\ \left(\frac{4-4n}{3}\right)\omega(n,m)+\left(\frac{2n-1}{3}\right) \\ \quad\times\left(\frac{\omega(n+1,m-1)}{2}-\omega(n+1,m-2)+\frac{\omega(n+1,m-3)}{2}\right) \\ \quad\text{otherwise}\end{cases}$$

Setting $\kappa(n,m)=0$ implies that

$$\omega(n,m)=\begin{cases}\frac{1}{3} & \text{if } n=0 \text{ and } m\geqslant 1 \\ \frac{4(m+1)(m+2)}{3} & \text{if } n=1 \text{ and } m\geqslant 0 \\ 0 & \text{otherwise}\end{cases}$$

By summing the series which defines $u(s,t)$,

$$u(s,t)=\frac{(1-t-2s)\,t}{3(1-t)}+\frac{(1-s-2t)\,s}{3(1-s)}+\frac{8(1-t-2s)\,st(1-s-t)}{3(1-t)^3}+\frac{8(1-s-2t)\,st(1-s-t)}{3(1-s)^3} \tag{7.12}$$

If one generalizes this approach and repeats the procedure for $d=4$, the solution is given by

$$u(r,s,t)=\frac{(1-r-s-2t)(r+s)}{8(1-r-s)}+\frac{(1-r-t-2s)(r+t)}{8(1-r-t)}$$
$$+\frac{(1-s-t-2r)(s+t)}{8(1-s-t)}$$
$$+\frac{(1-r-s-t)(1-r-s-2t)(r+s)\,t}{(1-r-s)^3}$$
$$+\frac{(1-r-s-t)(1-r-t-2s)(r+t)\,s}{(1-r-t)^3}$$
$$+\frac{(1-r-s-t)(1-s-t-2r)(s+t)\,r}{(1-s-t)^3} \tag{7.13}$$

Comparing the $d=3$ and $d=4$ solutions suggests a pattern. Since computing the coefficients is complicated in general, it is more convenient to verify that the candidate solution satisfies (7.10).

7.3. The Solution to the Continuous Problem

In this section, the pattern suggested by (7.12) and (7.13) is shown to satisfy (7.10). The proof itself heavily exploits the structure of the solution and thus reveals the properties of the solution which enable it to satisfy (7.10).

Definition 3. For $u\colon \mathbb{S}^{d-1}\to\mathbb{R}$, u is homogeneous with respect to L_d if $L_d u=0$. Denote the set of all symmetric functions which are homogeneous with respect to L_d by $\mathscr{H}_d$.

Proposition 5 is an immediate consequence of Definition 3.

Proposition 5. If $u\in\mathscr{H}_d$ and $T_d(u+1/d)-1/d\in\mathscr{H}_d$, then $v=u+1/d$ is a symmetric solution of (7.10).

Let φ be the projection of $\mathbb{S}^{d-1}$ onto $\mathbb{S}^2$ defined by

$$\varphi(s_1, s_2,\dots, s_{d-1})=(s_1, s_2+\cdots+s_{d-1}) \tag{7.14}$$

Given a function $f\colon \mathbb{S}^2 \to \mathbb{R}$, let $S_d f$ be the symmetrized extension of f to $\mathbb{S}^{d-1}$ defined by

$$S_d f(s_1,..., s_{d-1}) = \sum_{i=1}^{d-1} f \circ \varphi(s_i, s_1,..., s_{i-1}, s_{i+1},..., s_{d-1})$$

It is immediate that S_d is a linear operator and that $S_d f$ is symmetric. The class of functions that will be considered here are all symmetrized extensions. In particular, we consider $u \in \mathscr{H}_d$ such that $u = S_d f$ some $f\colon \mathbb{S}^2 \to \mathbb{R}$. By restricting attention to this class, we can view our solution as a sum of functions of two variables. There is a simple criterion for functions $f\colon \mathbb{S}^2 \to \mathbb{R}$ which implies that $S_d f \in \mathscr{H}_d$.

Definition 4. Given $f\colon \mathbb{S}^2 \to \mathbb{R}$, we say that f is *cancelative* if $f(s, t) + f(1 - s - t, t) = 0$ for all $(s, t) \in \mathbb{S}^2$.

Proposition 6. If $f\colon \mathbb{S}^2 \to \mathbb{R}$ is cancelative, then $S_d f \in \mathscr{H}_d$.

Proof. By definition, $s_1 + \cdots + s_d = 1$. Thus, for $1 \leqslant i < j \leqslant d$,

$$\begin{aligned}
&\varphi_1(s_i, s_1,..., s_{i-1}, s_{i+1},..., s_{j-1}, s_{j+1},..., s_d)\\
&\quad = 1 - \varphi_1(s_j, s_1,..., s_{i-1}, s_{i+1},..., s_{j-1}, s_{j+1},..., s_d)\\
&\qquad - \varphi_2(s_j, s_1,..., s_{i-1}, s_{i+1},..., s_{j-1}, s_{j+1},..., s_d)\\
&\varphi_2(s_i, s_1,..., s_{i-1}, s_{i+1},..., s_{j-1}, s_{j+1},..., s_d)\\
&\quad = \varphi_2(s_j, s_1,..., s_{i-1}, s_{i+1},..., s_{j-1}, s_{j+1},..., s_d)
\end{aligned}$$

where φ_i denotes the ith coordinate of φ. Combining (7.15) with the fact that f is cancelative implies that

$$\begin{aligned}
&f \circ \varphi(s_i, s_1,..., s_{i-1}, s_{i+1},..., s_{j-1}, s_{j+1},..., s_d)\\
&\quad + f \circ \varphi(s_j, s_1,..., s_{i-1}, s_{i+1},..., s_{j-1}, s_{j+1},..., s_d) = 0
\end{aligned} \tag{7.16}$$

By definition,

$$\begin{aligned}
L_d S_d f &= \sum_{i=1}^{d} S_d f(s_1,..., s_{i-1}, s_{i+1},..., s_d)\\
&= \sum_{i=2}^{d} \sum_{j<i} f \circ \varphi(s_j,..., s_{j-1}, s_1,..., s_{i-1}, s_{i+1},..., s_d)\\
&\quad + \sum_{i=1}^{d-1} \sum_{i<j} f \circ \varphi(s_j,..., s_{i-1}, s_{i+1},..., s_1, s_{j+1},..., s_d)
\end{aligned}$$

By switching the order of the second pair of summations, (7.16) implies that

$$
\begin{aligned}
L_d S_d f &= \sum_{i=2}^{d} \sum_{j<i} f \circ \varphi(s_j, ..., s_{j-1}, s_1, ..., s_{i-1}, s_{i+1}, ..., s_d) \\
&\quad + \sum_{i=2}^{d} \sum_{j<i} f \circ \varphi(s_i, ..., s_{j-1}, s_{j+1}, ..., s_1, s_{i+1}, ..., s_d) \\
&= \sum_{i=2}^{d} \sum_{j<i} (f \circ \varphi(s_j, ..., s_{j-1}, s_1, ..., s_{i-1}, s_{i+1}, ..., s_d) \\
&\quad + f \circ \varphi(s_i, ..., s_{j-1}, s_{j+1}, ..., s_1, s_{i+1}, ..., s_d)) \\
&= 0
\end{aligned}
$$

completing the proof. □

Two examples of cancelative functions are

$$(1-t-2s) \qquad \text{and} \qquad s(1-s-t)(1-t-2s)$$

These two examples will be the main building blocks for the solution to (7.10). Note that if either example is multiplied by a function which depends only on the variable t, then the resulting function is also cancelative. In particular, if

$$f(s, t) = \frac{(1-t-2s)\,t}{(1-t)} \qquad \text{and} \qquad g(s, t) = \frac{(1-t-2s)\,s(1-s-t)\,t}{(1-t)^3} \tag{7.17}$$

then $S_d f$ and $S_d g$ are elements of $\mathscr{H}_d$. Furthermore, $S_d(a_d f + b_d g)$ is an element of $\mathscr{H}_d$ for any real constants a_d and b_d. Our goal is to choose a_d and b_d such that $T_d(a_d f + b_d g + 1/d) - 1/d \in \mathscr{H}_d$.

Proposition 7. For all $f: \mathbb{S}^2 \to \mathbb{R}$, $T_d(f \circ \varphi(s_1, ..., s_{d-1})) = (T_3 f) \circ \varphi(s_1, ..., s_{d-1})$. In particular, $T_d S_d f = S_d T_3 f$.

Proof. By the chain rule,

$$s_1 \frac{\partial}{\partial s_1} f \circ \varphi(s_1, ..., s_{d-1}) = s_1 \frac{\partial f}{\partial s}(\varphi(s_1, ..., s_{d-1}))$$

$$s_i \frac{\partial}{\partial s_i} f \circ \varphi(s_1, ..., s_{d-1}) = s_i \frac{\partial f}{\partial t}(\varphi(s_1, ..., s_{d-1})) \qquad i \neq 1$$

Therefore,

$$\sum_{i=1}^{d-1} s_i \frac{\partial}{\partial s_i} f \circ \varphi(s_1,..., s_{d-1})$$

$$= \varphi_1(s_1,..., s_{d-1}) \frac{\partial f}{\partial s} (\varphi(s_1,..., s_{d-1})) + \varphi_2(s_1,..., s_{d-1}) \frac{\partial f}{\partial t} (\varphi(s_1,..., s_{d-1}))$$

completing the proof. □

As a consequence of Proposition 7 and linearity of both S_d and T_d,

$$T_d S_d (a_d f + b_d g) = a_d S_d T_3 f + b_d S_d T_3 g$$

Therefore, it is enough to compute $T_3 f$ and $T_3 g$. In light of Proposition 6, the next objective is to collect all cancelative parts of $T_3 f$ and $T_3 g$.

Proposition 8. If f and g are defined by (7.17), then

$$T_3 f(s, t) = \frac{4t}{3(1-t)} - \frac{t}{1-s-t} + \frac{2}{3}\left(2 + \frac{t}{1-t}\right) f(s, t) \tag{7.18}$$

$$T_3 g(s, t) = \frac{-st}{3(1-t)^2} + \frac{2}{3}\left(4 + \frac{3t}{1-t}\right) g(s, t) \tag{7.19}$$

Proof. We have

$$s \frac{\partial f}{\partial s}(s, t) = \frac{-2st}{1-t} \tag{7.20}$$

$$t \frac{\partial f}{\partial t}(s, t) = \frac{t(1-t-2s)}{1-t} + \frac{-t^2}{1-t} + \frac{t^2(1-t-2s)}{(1-t)^2} \tag{7.21}$$

The first term in (7.21) is $f(s, t)$. By combining the second term in (7.21) with (7.21) and adding and subtracting $t/(1-t)$, another copy of $f(s, t)$ can be obtained. The final term in (7.21) is simply $f(s, t)$ scaled by a function which depends only on the variable t. Thus,

$$s \frac{\partial f}{\partial s}(s, t) + t \frac{\partial f}{\partial t}(s, t) = \left(2 + \frac{t}{1-t}\right) f(s, t) - \frac{t}{1-t}$$

Observing that

$$\frac{f(s,t)}{1-s-t}=\frac{2t}{1-t}-\frac{t}{(1-s-t)}$$

gives

$$T_3 f(s,t)=\frac{2t}{1-t}-\frac{t}{(1-s-t)}+\frac{2}{3}\left(\left(2+\frac{t}{1-t}\right)f(s,t)-\frac{t}{1-t}\right)$$
$$=\frac{4t}{3(1-t)}-\frac{t}{(1-s-t)}+\frac{2}{3}\left(2+\frac{t}{1-t}\right)f(s,t)$$

establishing (7.18).

For $g(s,t)$,

$$s\frac{\partial g}{\partial s}(s,t)=\frac{s(1-t-2s)(1-s-t)\,t}{(1-t)^3}+\frac{-2s^2(1-s-t)\,t}{(1-t)^3}+\frac{-s^2(1-t-2s)\,t}{(1-t)^3} \tag{7.22}$$

$$t\frac{\partial g}{\partial t}(s,t)=\frac{-ts(1-s-t)\,t}{(1-t)^3}+\frac{-t(1-t-2s)\,st}{(1-t)^3}+\frac{t(1-t-2s)\,s(1-s-t)}{(1-t)^3}$$
$$+\frac{3t(1-t-2s)\,s(1-s-t)\,t}{(1-t)^4} \tag{7.23}$$

In a similar manner as with $f(s,t)$, combine the second and third terms in (7.22) with the first and second terms in (7.23) respectively to obtain

$$s\frac{\partial g}{\partial s}(s,t)+t\frac{\partial g}{\partial t}(s,t)=\left(4+\frac{3t}{1-t}\right)g(s,t)-\frac{s(2-2t-3s)\,t}{(1-t)^3}$$

Since

$$\frac{g(s,t)}{1-s-t}-\frac{2s(2-2t-3s)\,t}{3(1-t)^3}=\frac{-st}{3(1-t)^2}$$

(7.19) holds. □

Proposition 8 decomposes $T_3 f$ and $T_3 g$ into cancelative and noncancelative components. Denote the noncancelative terms by

$$\varepsilon_1(s,t)=\frac{4t}{3(1-t)},\qquad \varepsilon_2(s,t)=\frac{-t}{1-s-t},\qquad \text{and}\qquad \varepsilon_3(s,t)=\frac{-st}{3(1-t)^2}$$

Recall our ultimate goal, to choose a_d and b_d such that $T_d S_d(a_d f + b_d g) + T_d(1/d) - 1/d$ is an element of $\mathcal{H}_d$. Since

$$T_d(1/d) - 1/d = \frac{s_1 + \cdots + s_{d-1}}{d(1 - s_1 - \cdots - s_{d-1})}$$

and

$$S_d \varepsilon_2 = \frac{-(d-2)(s_1 + \cdots + s_{d-1})}{1 - s_1 - \cdots - s_{d-1}} \tag{7.24}$$

it s natural to choose a_d such that $S_d a_d \varepsilon_2$ cancels $T_d(1/d) - 1/d$. In particular, $a_d = 1/((d-2)\, d)$. With only ε_1 and ε_3 remaining, b_d is chosen such that $a_d \varepsilon_1 + b_d \varepsilon_3$ is cancelative. Setting $b_d = 8a_d$ gives

$$a_d \varepsilon_1(s, t) + 8a_d \varepsilon_3(s, t) = a_d \frac{4t(1 - t - 2s)}{3(1-t)^2} = a_d \frac{4}{3(1-t)} f(s, t) \tag{7.25}$$

which is cancelative.

Theorem 8. Let $h(s, t) = f(s, t) + 8g(s, t)$ where f and g are defined by (7.17). Then $S_d a_d h + 1/d$ is a symmetric solution to (7.10).

Proof. Since $h(s, t)$ is cancelative, Proposition 6 implies that $S_d a_d h \in \mathcal{H}_d$. By Proposition 8 and equation (7.25),

$$T_3 a_d h(s, t) = a_d \frac{2(4-t)}{3(1-t)} h(s, t) + a_d \varepsilon_2(s, t) \tag{7.26}$$

Since $h(s, t)$ is cancelative, Propositions 6, 7, and (7.24) imply that $T_d(S_d a_d h + 1/d) - 1/d$ is an element of $\mathcal{H}_d$. By Proposition 5, the assertion holds. □

Theorem 9. $S_d a_d h + 1/d$ is absolutely bounded by one.

Proof. We have

$$\frac{\partial h}{\partial s}(s, t) = \frac{6t((1-t)^2 - 8(1-t)\, s + 8s^2)}{(1-t)^3}$$

Therefore, the maximum and minimum occur at

$$s_{\max} = \frac{(1-t)(2-\sqrt{2})}{4} \qquad \text{and} \qquad s_{\min} = \frac{(1-t)(2+\sqrt{2})}{4}$$

respectively. Since

$$h\left(\frac{(1-t)(2-2\sqrt{2})}{4}, t\right)=\sqrt{2}\, t$$

and

$$h\left(\frac{(1-t)(2+2\sqrt{2})}{4}, t\right)=-\sqrt{2}\, t$$

It follows that

$$|h\circ\varphi(s_1,..., s_{d-1})| \leqslant \sqrt{2} \qquad \text{on} \qquad \mathbb{S}^{d-1}$$

Since $S_d h$ has $d-1$ terms of the form $h\circ\varphi$,

$$\left|S_d a_d h+\frac{1}{d}\right| \leqslant \frac{(d-1)\sqrt{2}}{(d-2)\, d}+\frac{1}{d}$$

which is bounded by one provided $d \geqslant 4$. Since $S_3 h(s, t)=h(s, t)+h(t, s)$, it is possible to use the better bound of

$$|h(s, t)+h(t, s)| \leqslant \sqrt{2}\,(s+t) \leqslant \sqrt{2}$$

Thus,

$$\left|S_3 a_3 h+\frac{1}{3}\right| \leqslant \frac{\sqrt{2}+1}{3}$$

as desired. □

Theorem 2 follows from Theorems 8, 9, and Proposition 8. Presumably, a suitably bounded solution to the discrete problem exists which has a structure analogous to the structure of the solution to the partial differential equation. Our attempts to exploit this structure have failed. Nevertheless, we believe that (6.19) and (6.22) has a solution for all $n \in \mathbb{N}$.

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