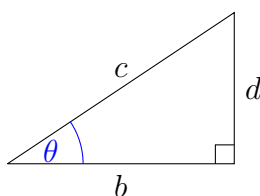


Trigonometric Substitutions

Expression	Substitution	Domain	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

Trigonometric Substitution Diagram

When solving a problem with trigonometric substitution, we may need to switch back to having things in terms of x . A triangle like the one below can help us.



Using the equation from our substitution, we can fill in our triangle.

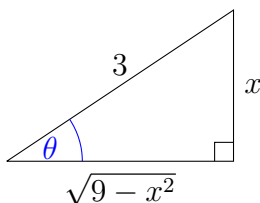
Example 1

$$\int \frac{\sqrt{9-x^2}}{x^2} dx$$

This is of the form $\sqrt{a^2 - x^2}$, so we let $x = 3 \sin \theta$. Then $dx = 3 \cos \theta d\theta$.

$$\begin{aligned} \int \frac{\sqrt{9-x^2}}{x^2} dx &= \int \frac{\sqrt{9-(3 \sin \theta)^2}}{(3 \sin \theta)^2} (3 \cos \theta) d\theta = \int \frac{\sqrt{9-9 \sin^2 \theta}}{9 \sin^2 \theta} (3 \cos \theta) d\theta = \int \frac{3 \sqrt{1-\sin^2 \theta}}{9 \sin^2 \theta} (3 \cos \theta) d\theta \\ &= \int \frac{\sqrt{1-\sin^2 \theta}}{\sin^2 \theta} (\cos \theta) d\theta = \int \frac{\cos \theta}{\sin^2 \theta} (\cos \theta) d\theta = \int \frac{\cos^2 \theta}{\sin^2 \theta} d\theta = \int \cot^2 \theta d\theta = \int (\csc^2 \theta - 1) d\theta \\ &= -\cot \theta - \theta + C \end{aligned}$$

Next we need to have everything in terms of x . $x = 3 \sin \theta$ tells us $\sin \theta = \frac{x}{3}$. Also, $\theta = \arcsin \frac{x}{3}$. Now we can use a triangle to solve.



By our triangle, we know $\cot \theta = \frac{\sqrt{9-x^2}}{x}$.

Finally we can use substitution.

$$-\cot \theta - \theta + C = -\frac{\sqrt{9-x^2}}{x} - \arcsin \frac{x}{3} + C$$



Example 2

$$\int \frac{1}{x^2\sqrt{x^2+4}} dx$$

This fits the form of $\sqrt{a^2 + x^2}$. We let $x = 2 \tan \theta$. Then $dx = 2 \sec^2 \theta d\theta$. Next we substitute.

$$\int \frac{1}{(2 \tan \theta)^2 \sqrt{(2 \tan \theta)^2 + 4}} 2 \sec^2 \theta d\theta = \int \frac{1}{(4 \tan^2 \theta) 2 \sqrt{\tan^2 \theta + 1}} 2 \sec^2 \theta d\theta = \int \frac{1}{(4 \tan^2 \theta) 2 \sec \theta} 2 \sec^2 \theta d\theta$$

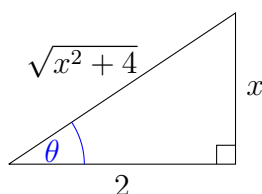
$$\frac{1}{4} \int \frac{1}{(\tan^2 \theta)} \sec \theta d\theta = \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} d\theta$$

Next we need to use u-substitution. Let $u = \sin \theta$, then $du = \cos \theta d\theta$.

$$\frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} d\theta = \frac{1}{4} \int \frac{1}{u^2} du = -\frac{1}{4} \frac{1}{u} + C$$

And by our u-substitution we have $-\frac{1}{4} \frac{1}{\sin \theta} + C$, or $-\frac{1}{4} \csc \theta + C$.

Next we use a triangle to put our answer in terms of x . By above, $\tan \theta = \frac{x}{2}$.



By our triangle, $-\frac{1}{4} \csc \theta + C = -\frac{\sqrt{x^2+4}}{4x} + C$.

Example 3

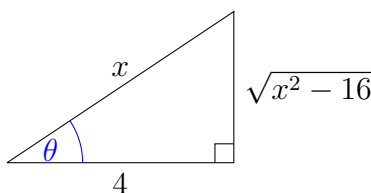
$$\int \frac{\sqrt{x^2-16}}{x} dx$$

This problem fits the last form, $\sqrt{x^2 - a^2}$. We let $x = 4 \sec \theta$, then $dx = 4 \sec \theta \tan \theta d\theta$.

$$\int \frac{\sqrt{(4 \sec \theta)^2 - 16}}{4 \sec \theta} 4 \sec \theta \tan \theta d\theta = \int \sqrt{(4 \sec \theta)^2 - 16} \cdot (\tan \theta) d\theta = \int 4 \tan \theta (\tan \theta) d\theta = \int 4 \tan^2 \theta d\theta$$

$$= 4 \int (\sec^2 \theta - 1) d\theta = 4 \tan \theta - 4\theta + C$$

By above, we have $\sec \theta = \frac{x}{4}$, $\theta = \sec^{-1}(\frac{x}{4})$, and the following:



Then our answer becomes $4 \frac{\sqrt{x^2-16}}{4} - 4 \sec^{-1}(\frac{x}{4}) + C = \sqrt{x^2 - 16} - 4 \sec^{-1}(\frac{x}{4}) + C$

