

Method of Cylindrical Shells is used when it becomes complicated to compute inner and outer radii of a washer. For example in Figure 1, we must solve for x in terms of y .

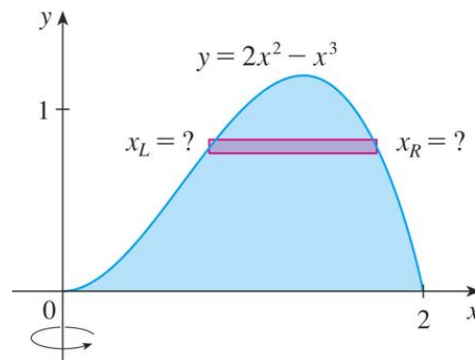


Figure 1

Figure 2 shows one cylindrical shell with inner radius r_1 , outer radius r_2 , and height h .

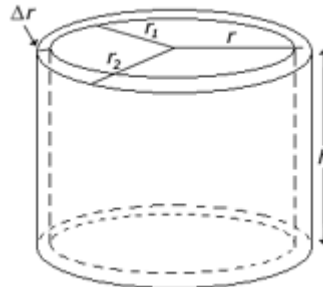


Figure 2

Its volume V is calculated as follows $V = V_2 - V_1$, where V_1 is a volume of the cylinder with inner radius r_1 and V_2 is a volume of the cylinder with outer radius r_2 . Remember that the volume of a cylinder is $V = \pi r^2 h$. So, we have the following formula for the volume of one cylinder

$$V = \pi r_2^2 h - \pi r_1^2 h = \pi(r_2^2 - r_1^2)h = \pi(r_2 + r_1)(r_2 - r_1)h$$

Observe that $r = \frac{r_2 + r_1}{2}$. So by multiplying and dividing the previous equation by 2 we get

$$V = 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1)$$

$$\mathbf{V = 2\pi r h \Delta r}$$

Now let S be the solid obtained by rotating about y -axis the region bounded by $y = f(x)$ where $f(x) \geq 0$, $y = 0$, $x = a$, and $x = b$, where $b > a \geq 0$.

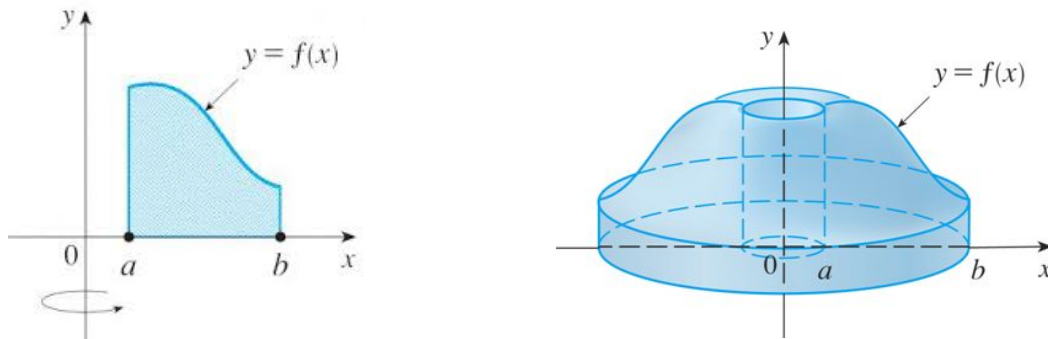


Figure 3

To obtain this volume, we divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width Δx and let \bar{x}_i be the midpoint of the i -th subinterval.

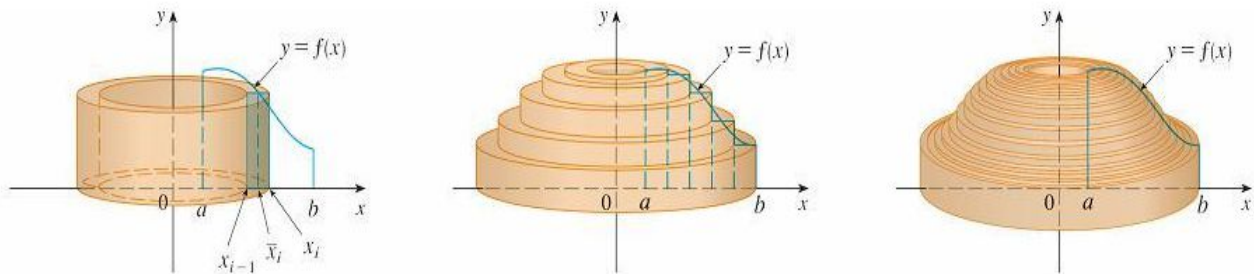


Figure 4

Then the volume of the solid in Figure 3, obtained by rotating about the y -axis the region under the curve $y = f(x)$ from a to b , is

$$V = \int_a^b 2\pi x f(x) dx \quad \text{where } 0 \leq a < b$$

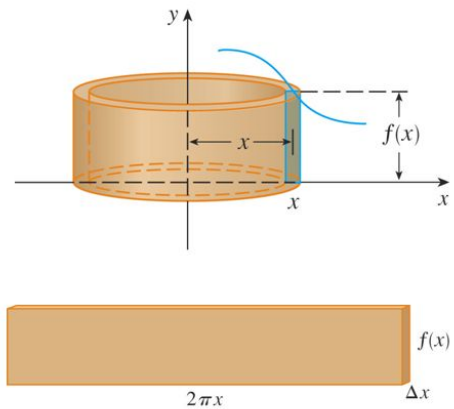


Figure 5

In Figure 5, we cut and flattened a typical cylindrical shell. So we have

$$\int_a^b (2\pi x) [f(x)] dx$$

circumference *height* *thickness*

Observe that x represents the radius of each cylinder when we rotate the region under a curve about the y -axis. If we revolve about the x -axis, then the radius is y .

Example 1 Find the volume of the solid obtained by rotating about the y -axis the region between $y = x$ and $y = x^2$

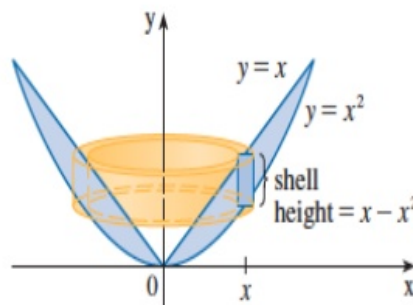


Figure 6

Solution:

The shell has radius x , circumference $2\pi x$, and the height $x - x^2$. So, the volume is

$$V = \int_0^1 (2\pi x)(x - x^2) dx = 2\pi \int_0^1 (x^2 - x^3) dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{\pi}{6}$$

Example 2 Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

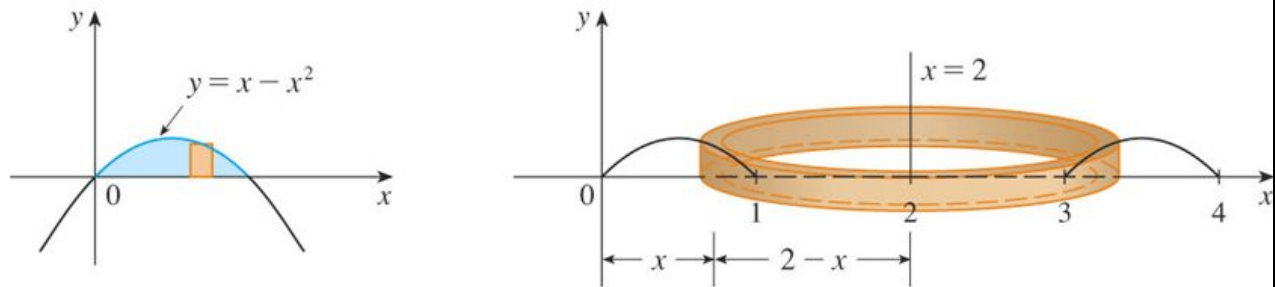


Figure 7

Solution:

Figure 7 shows the region and cylindrical shell formed by rotation about the line $x = 2$. It has radius $2 - x$, circumference $2\pi(2 - x)$, and height $x - x^2$.

The volume of the given solid is

$$V = \int_0^1 2\pi(2 - x)(x - x^2) dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x) dx = 2\pi \left[\frac{x^4}{4} - x^3 + x^2 \right]_0^1 = \frac{\pi}{2}$$