A. The degree of the numerator is *greater than* the degree of the denominator.

1) Perform long division.
2) Integrate each term.

Example:

$$\int \frac{x^3 + x}{x - 1} dx$$

After performing long division and integration, we get

$$\int \frac{x^3 + x}{x - 1} dx = \int (x^2 + x + 2 + \frac{2}{x - 1}) dx = \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2ln|x - 1| + C$$

B. The degree of the numerator is *less than* the degree of the denominator.

Let's consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. Then

Factor in denominator $\boldsymbol{Q(x)}$			Term in partial fraction decomposition
1	Product of distinct linear factors	$(a_1x + b_1)(a_2x + b_2) \dots (a_kx + b_k)$	$\frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$
2	Product of linear factors, some of which are repeated	$(ax + b)^k$	$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_k}{(a_1x + b_1)^k}$
3	Irreducible quadratic factors, none of which is repeated	$ax^2 + bx + c$	$\frac{Ax + B}{ax^2 + bx + c}$

Factor in denominator $Q(x)$			Term in partial fraction decomposition
4	Repeated irreducible quadratic factor	$(ax^2+bx+c)^k$	$\frac{A_1+B_1}{ax^2+bx+c} + \frac{A_2+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_2+B_k}{(ax^2+bx+c)^k}$

Examples:

1. **Product of distinct linear factors.**

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$$

Factor the denominator.

$2x^3+3x^2-2x = x(2x^2+3x-2) = x(2x-1)(x+2)$

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

To determine the values of A, B, and C, we multiply both sides of the equation by the product of the denominators, $x(2x-1)(x+2)$, obtaining

$$\begin{aligned} x^2+2x-1 &= A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1) \\ &= A(2x^2+3x-2) + Bx^2 + 2Bx + 2Cx^2 - Cx \\ &= 2Ax^2 + 3Ax - 2A + Bx^2 + 2Bx + 2Cx^2 - Cx \\ &= (2A+B+2C)x^2 + (3A+2B-C)x - 2A \end{aligned}$$

Since the left hand side and the right hand side of the equation are equal, so coefficients must be equal. Thus,

$2A+B+2C = 1$, $3A+2B-C = 2$, and $-2A = -1$

Solving, we get $A = \frac{1}{2}, B = \frac{1}{5}, C = -\frac{1}{10}$, and so

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx = \int \left[\frac{1}{2}\left(\frac{1}{x}\right) + \frac{1}{5}\left(\frac{1}{2x-1}\right) - \frac{1}{10}\left(\frac{1}{x+2}\right)\right]dx = \frac{1}{2}ln|x| + \frac{1}{10}ln|2x-1| - \frac{1}{10}ln|x+2| + K$$

2. **Product of linear factors, some of which are repeated.**

$$\int \frac{4x}{x^3 - x^2 - x + 1} dx$$

Factor the denominator.

$$x^3 - x^2 - x + 1 = x^2(x-1) - (x-1) = (x^2-1)(x-1) = (x-1)(x+1)(x-1)$$
$$= (x-1)^2(x+1)$$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2 = (A+C)x^2 + (B-2C)x + (-A+B+C)$$

Comparing coefficients, we get

$A + C = 0$, $B - 2C = 4$, and $-A + B + C = 0$

Solving, we obtain $A = 1, B = 2, C = -1$, so

$$\int \frac{4x}{x^3 - x^2 - x + 1} dx = \int \frac{4x}{(x-1)^2(x+1)} dx = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\right) dx$$
$$= ln|x-1| - \frac{2}{x-1} - ln|x+1| + K = ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + K$$

3. **Irreducible quadratic factors, none of which is repeated.**

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$$

Since $x^3 + 4x = x(x^2 + 4)$ can't be factored further, we write

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x = (A + B)x^2 + Cx + 4A$$

So, $A + B = 2, C = -1, 4A = 4$, thus $A = 1, B = 1, C = -1$ and thus

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$$
$$= \int \frac{2x^2 - x + 4}{x(x^2+4)} dx$$
$$= \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right) dx = \int \frac{1}{x} dx + \int \frac{x}{x^2+4} dx - \int \frac{1}{x^2+4} dx$$

$$= ln|x| + \frac{1}{2}ln(x^2+4) - \frac{1}{2}tan^{-1}\left(\frac{x}{2}\right) + K$$

4. Repeated irreducible quadratic factor.

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}dx$$

$$\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\begin{aligned} -x^3 + 2x^2 - x + 1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A \end{aligned}$$

Comparing coefficients, we get $A + B = 0, C = -1, 2A + B + D = 2, C + E = -1, A = 1$, then $A = 1, B = -1, C = -1, D = 1, E = 0$. Thus

$$\begin{aligned} &\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}dx \\ &= \int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx \\ &= \int \frac{1}{x}dx - \int \frac{x}{x^2+1}dx - \int \frac{1}{x^2+1}dx + \int \frac{x}{(x^2+1)^2}dx \\ &= ln|x| - \frac{1}{2}ln(x^2+1) - tan^{-1}x - \frac{1}{2(x^2+1)} + K \end{aligned}$$