



### Squeeze Theorem pg 101 Stewart 8th edition

Given a function  $f(x)$  such that when  $x$  is arbitrarily close to  $a$  then  $g(x) \leq f(x) \leq h(x)$  :

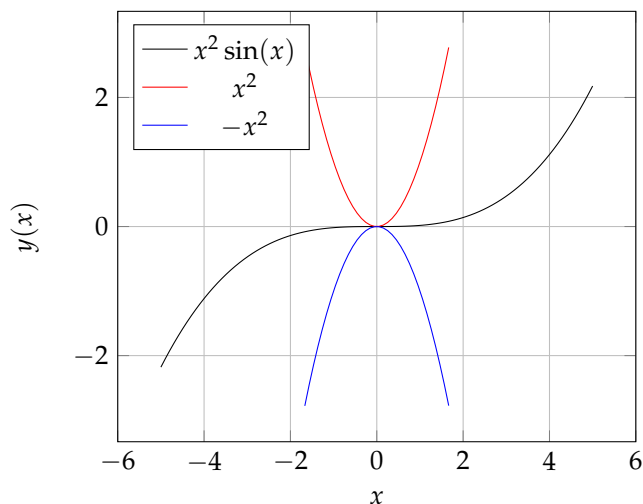
Example of alternating/oscillating function:  $-1 \leq \sin(x) \leq 1$

If we want to find the  $\lim_{x \rightarrow a} f(x)$

then we can show that the limits of its bounds  $g(x)$  and  $h(x)$  are equal

1. So if we show that  $\lim_{x \rightarrow a} g(x) = L$  and  $\lim_{x \rightarrow a} h(x) = L$
2. then since  $g(x) \leq f(x) \leq h(x)$ , when  $x$  is close to  $a$  then  $L \leq \lim_{x \rightarrow a} f(x) \leq L$
3. So then we can conclude that  $\lim_{x \rightarrow a} f(x) = L$  by the squeeze theorem

**Example:** Find  $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^2 \sin(x)$



1. Establish your bounds: We know that  $-1 \leq \sin(x) \leq 1$  so  $-x^2 \leq x^2 \sin(x) \leq x^2$
2. Take the limit of the endpoints:  $\lim_{x \rightarrow 0} -x^2 = 0$  and  $\lim_{x \rightarrow 0} x^2 = 0$
3. Conclude: So when  $x$  is arbitrarily close to 0 then  $0 \leq \lim_{x \rightarrow 0} x^2 \sin(x) \leq 0$   
so then  $\lim_{x \rightarrow 0} x^2 \sin(x) = 0$  by the squeeze theorem

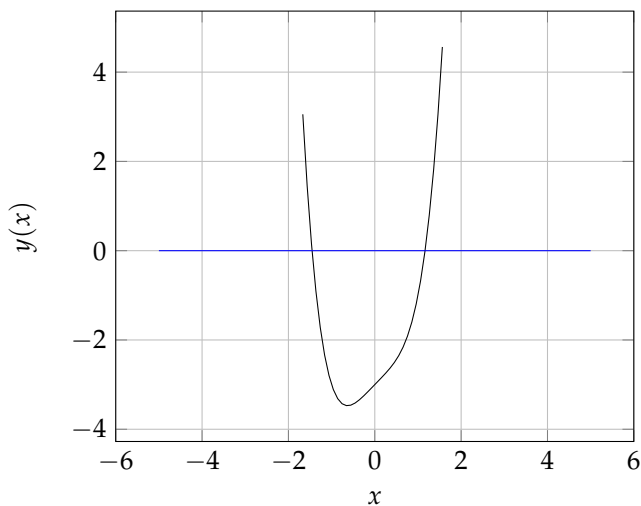




### Intermediate Value Theorem pg 122 Stewart 8th edition

1. Show  $f(x)$  is continuous on the closed interval  $[a, b]$
2. let  $N$  be a number between  $f(a)$  And  $f(b)$  where  $f(a) \neq f(b)$
3. Then there exists a number  $c$  in  $(a, b)$  such that  $f(c) = N$   
\*This is commonly used to show that there exists a root of an equation in a certain interval, Let  $N = 0$ \*

**Example:** Show that there is a root to  $f(x) = x^4 + x - 3$  in the interval  $(1, 2)$



1.  $f(x) = x^4 + x - 3$  is a polynomial therefore continuous on  $[1, 2]$
2. Plug in endpoints:  $f(1) = -1$ , which is negative  $f(1) < 0$  and  $f(2) = 15$ , which is positive  $f(2) > 0$
3. Since  $f(1) < 0$  and  $f(2) > 0$  there exists a point  $c$  in  $(1, 2)$  such that  $f(c) = 0$





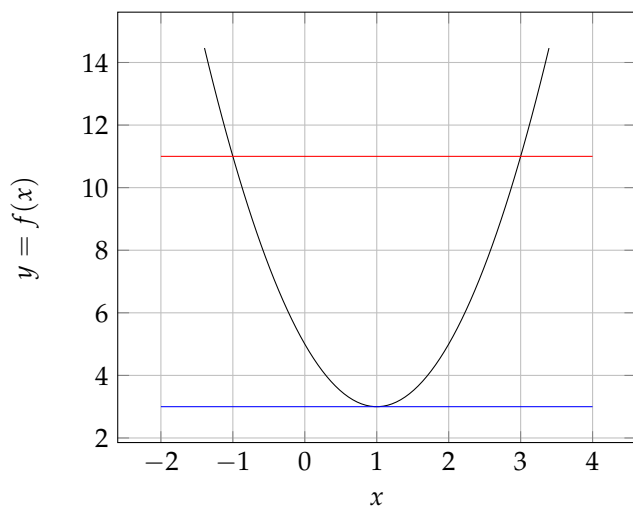
# Calculus

## Theorems

### Rolle's Theorem pg 287 Stewart 8th edition

1. Show  $f(x)$  is continuous on the closed interval  $[a, b]$   
(that is find if there are any points of discontinuity in the given interval)
2. Show that  $f(x)$  is differentiable on the open interval  $(a, b)$   
(continuous with no corners or cusps or any other immediate changes in slope)
3. Show  $f(a) = f(b)$  then you can conclude there exists a number  $c$  in  $(a, b)$  such that  $f'(c) = 0$

**Example:** Verify that there exists a critical point of  $f(x) = 2x^2 - 4x + 5$  on the interval  $[-1, 3]$



1.  $f(x)$  is a polynomial therefore continuous on  $(-\infty, \infty)$
2.  $f'(x) = 4x - 4$  exists for all  $x$  in  $[-1, 3]$ , hence  $f(x)$  differentiable
3.  $f(-1) = 11$  and  $f(3) = 11$  so  $f(-1) = f(3)$   
so there exists a  $c$  in  $(-1, 3)$  such that  $f'(c) = 0$  by Rolle's Theorem.
4. Find such a  $c$  such that  $f'(c) = 0$ , so  $f'(x) = 4x - 4$ .

$$f'(x) = 0 \tag{1}$$

$$4x - 4 = 0 \tag{2}$$

$$4x = 4 \tag{3}$$

$$x = 1 \tag{4}$$

So  $c = 1$ , and  $f'(c) = 4(1) - 4 = 4 - 4 = 0$  ✓ Observe that  $x=1$  is in the interval  $(-1, 3)$

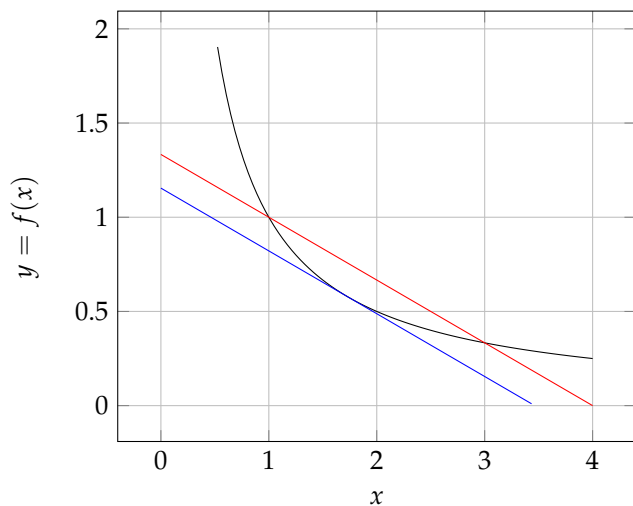




### Mean Value Theorem pg 288 Stewart 8th edition

1. Show  $f(x)$  is continuous on the closed interval  $[a, b]$
2. Show  $f(x)$  is differentiable on the open interval  $(a, b)$
3. Then there exists a number  $c$  in  $(a, b)$  such that  $f'(c) = \frac{f(b)-f(a)}{b-a}$  that is the tangent line to  $(c, f(c))$  has the same slope as the secant through  $(a, f(a))$  and  $(b, f(b))$

**Example:** Show that there exists a point in the interval  $[1, 3]$  such that the slope of the tangent line to that point is equal to the slope of the secant line through it's endpoints, on the curve of  $f(x) = \frac{1}{x}$



1. The only discontinuity is at partition point  $x = 0$  which is not in  $[1, 3]$  so  $f(x)$  is continuous on  $[1, 3]$ .
2.  $f'(x) = \frac{-1}{x^2}$  is only discontinuous at partition point  $x = 0$  which is not in the interval  $[1, 3]$  so  $f(x)$  is differentiable on  $(1, 3)$   
Plug in the endpoints and find the slope through them:  $f(1) = 1, f(3) = \frac{1}{3}$  so  $\frac{f(3)-f(1)}{3-1} = \frac{-1}{3}$
3. Then there exists a  $c$  in  $[1, 3]$  such that  $f'(c) = \frac{-1}{3}$  so  $f'(c) = \frac{-1}{c^2} = \frac{-1}{3}$  solving we get that  $c = \pm\sqrt{3}$  since  $+\sqrt{3}$  is in the interval  $(1, 3)$  so  $c = +\sqrt{3}$



### L' Hospital's Rule pg 305 Stewart 8th edition

$$\text{L'Hospital's Rule: } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

If you are asked to take the limit of a rational equation of the form  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$

such that the limit is an **Indeterminate Form** or of the form  $\frac{\infty}{\infty}$  or  $\frac{0}{0}$

1. First take the limit of the rational function and show that it is an indeterminate form:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty} \text{ or } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

2. Then take the derivative of the numerator and the denominator separately and take the limit:

$$\text{Find } \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

3. If the limit goes to a number  $N$ ,  $0$ , or  $\infty$  then that is your answer.

if it gives another indeterminate form i.e.  $\frac{\infty}{\infty}$  or  $\frac{0}{0}$

then repeat the process until the limit goes to a number,  $0$  or  $\infty$ .

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**Example:**  $\lim_{x \rightarrow \infty} \frac{\ln(x)}{x}$

1. Show that the limit gives an indeterminate form:  $\lim_{x \rightarrow \infty} \frac{\ln(x)}{x} = \frac{\infty}{\infty}$
2. Take the derivative of the numerator and denominator separately.  $\frac{dy}{dx} \ln(x) = \frac{1}{x}$  and  $\frac{dy}{dx} x = 1$
3. Take the limit of  $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$   $\lim_{x \rightarrow \infty} \frac{1}{x} = \frac{1}{\infty} = 0$

