Fibonacci Numbers and Modular Arithmetic

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The Fibonacci Sequence start with $F_1 = 1$ and $F_2 = 1$. Then the two consecutive numbers are added to find the next term. The Lucas Sequence starts with $L_1 = 1$ and $L_2 = 2$ following the same rule of adding two previous consecutive numbers to find the next term.

Fibonacci Sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144... *Recursive Formula*: $F_n = F_{n-1} + F_{n-2}$

Lucas Sequence: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... *Recursive Formula*: $L_n = L_{n-1} + L_{n-2}$

Table 1

n	F_n	L_n
1	1	1
2	1	2
3	2	3
4	3	5
5	5	8
6	8	13
7	13	21
8	21	34
9	34	55
10	55	89
11	89	144
12	144	233
13	233	377

Using Table 1 or the list given, here is an example of how the pattern works. **Given** $F_1 = 1$, $F_2 = 1$

$F_3 = F_1 + F_2 = 1 + 1 = 2$

$F_4 = F_2 + F_3 = 1 + 2 = 3$

$F_5 = F_3 + F_4 = 2 + 3 = 5$

$F_6 = F_4 + F_5 = 3 + 5 = 8$

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}$$

$$\textit{Golden Ratio} : \varphi = \frac{1 + \sqrt{5}}{2}$$

Table 2(on the right) is a table of fractions each found by the following fraction: $\frac{F_n}{F_{n-1}}$ which are the *relative* sizes of the Fibonacci numbers.The relative sizes can each be rewritten as the following examples:

$\frac{F_2}{F_1} = \frac{1}{1} = 1$

$\frac{F_3}{F_2} = \frac{2}{1} = 1 + \frac{1}{1}$

$\frac{F_4}{F_3} = \frac{3}{2} = 1 + \frac{1}{1 + \frac{1}{1}}$

$\frac{F_5}{F_4} = \frac{5}{3} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}$

The ratios will continue the pattern and eventually approach the unending number called φ ("phi") whose precise value is then calculated as the *Golden Ratio* using the equation $\varphi = 1 + \frac{1}{\varphi}$.

Table 2

Fraction of Adjacent Fibonacci Numbers	Decimal Equivalent
$\frac{1}{1}$	1.0
$\frac{2}{1}$	2.0
$\frac{3}{2}$	1.5
$\frac{5}{3}$	1.666...
$\frac{8}{5}$	1.6
$\frac{13}{8}$	1.625
$\frac{21}{13}$	1.6153...
$\frac{34}{21}$	1.6190...
$\frac{55}{34}$	6.176...

Example: *Rabbits* Suppose you begin with a pair of baby rabbits, one male and one female. The rabbits have a 1 month gestation period(1 month being in the womb) and they can reproduce after 1 month of being born. Each pair reproduces another pair. Assume no pair ever dies. How many pairs of rabbits will exist in a particular month?

Pattern:

Time in Months	Start	1	2	3	4	5	6	7
Number of Pairs	1	1	2	3	5	8	13	21

Note: The pattern of number of pairs each month follows the Fibonacci sequence.

Time in Months	Start	1	2	3	4	5	6	7
Number of Pairs of Parents	0	0	1	1	2	3	5	8
Number of Pairs of New Babies	1	0	1	1	2	3	5	8
Number of Pairs of Adults	0	1	0	1	1	2	3	5
Total Number of Pairs	1	1	2	3	5	8	13	21

Each new month, the number of pairs of new baby equal the number of pair of parents of the previous month. Adults become parents and new babies become adults.*

*New babies refers to those just born. Adults are 1 month olds and ready to reproduce. Parent pairs are those who just gave birth.

Example: *New Patterns* Determine a simple formula for $(F_n)^2 + (F_{n+1})^2$

n	1	2	3	4	5	6	7	8
$(F_n)^2$	1	1	4	9	25	64	169	441
$(F_{n+1})^2$	1	4	9	25	64	169	441	1156
sum	2	5	13	34	89	233	610	1597

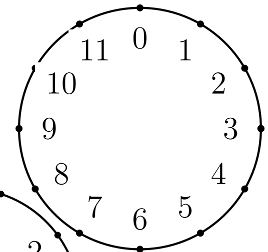
Note: The sum are all odd Fibonacci terms greater than F_1 (meaning F_3, F_5, etc) . Even numbers follow the pattern $2k$ while odd numbers follow the patten $2k+1$. The table helps identify a pattern that can be written as $(F_n)^2 + (F_{n+1})^2 = F_{2n+1}$ where n=1,2,3,4,...

Creating tables is a helpful method of identifying patterns that otherwise cannot immediately be seen.

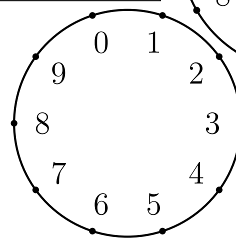
Modular Arithmetic(informally known as clock arithmetic): In modular arithmetic, numbers "wrap around" upon reaching a given fixed quantity, which is known as the modulus (which would be 12 in the case of hours on a clock). When working with 12 as the modulus, we can say we are working with *mod 12*

Equivalence: $\equiv$ means *equivalent* which is not the same as equal.

For example, on a modulus 12 clock, 12 is equivalent to 0; therefore, $12 \equiv 0 \bmod 12$ which can be read as *"12 is equivalent to 0 mod 12."* Another example is 37 is equivalent to 1. Start at 0 and count up to 37. You should return to 1. Therefore, $37 \equiv 1 \bmod 12$



Another Perspective Now suppose we are working with modulus 10. 10,20,30,40,50,60, etc are multiples of 10; therefore, they are all equivalent to 0 mod 10. 34 is a multiple of 10 with a remainder of 4; therefore,
$34 \equiv 4 \bmod 10$.
$a \equiv r \mod m$ a is an integer with a multiple of m with a remainder r.



Check Digits

The following formulas are used to verify identification numbers using modulus 10.

Bar Codes

$3d_1 + d_2 + 3d_3 + d_4 + 3d_5 + d_6 + 3d_7 + d_8 + 3d_9 + d_{10} + 3d_{11} + c \equiv 0 \bmod 10$

There are 12 digits and c is the check digit.

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$d_1 + 3d_2 + d_3 + 3d_4 + d_5 + 3d_6 + d_7 + 3d_8 + d_9 + 3d_{10} + d_{11} + 3d_{12} + d_{13} \equiv 0 \bmod 10$

The last digit, d_{13} , is the check digit.

Checks

$7n_1 + 3n_2 + 9n_3 + 7n_4 + 3n_5 + 9n_6 + 7n_7 + 3n_8 + 9n_9 \equiv 0 \mod 10$

The last digit, n_9 is the check digit.

Example: Given the bar code 3 4 0 0 3 2 6 9 1 2 0 c. Find the check digit c.

Step 1: Use the formula for bar codes: 3(3)+ 4 + 3(0)+ 0 + 3(3)+2 + 3(6) + 9 + 3(1)+ 2 + 3(0) = 56

Step 2: We need $56 + c \equiv 0 \mod 10$ which means we need a multiple of 10 and r=0. Note that 56 + 4 = 60

Step 3: $60 \equiv 0 \mod 10$. This tells us $c = 4$.

Fermet's Little Theorem If p is a prime number and n is any integer that does not have p as a factor then n^{p-1} is equivalent to $1 \mod p$. In other words, n^{p-1} will always have a remainder of 1 when divided by p. Notation: $n^{p-1} \equiv 1 \mod p$

Some Rules

If $a = qm + r$ then $a \equiv r \mod m$

If $a \equiv r \mod m$ then $a^k \equiv r^k \mod m$

$a \equiv r \mod m \Leftrightarrow a + b \equiv r + b \mod m$

$a \equiv r \mod m \Leftrightarrow a - b \equiv r - b \mod m$

$a \equiv r \mod m \Leftrightarrow ab \equiv rb \mod m$

Note: a,q,m,r,k are all integers and $\Leftrightarrow$ means it goes both ways

Simplifying Modulos

Example: Given: $5^6 \equiv r \mod 7$ Find r.

Step 1: Use *Fermet's Little Theorem.*

We know we are working with $p = 7$ and $p - 1 = 7 - 1 = 6$

Step 2: Confirm 5 does not have a factor of 7.

Therefore, $5^{7-1} \equiv 5^6 \equiv 1 \mod 7$

Example: Given $5^{600} \equiv r \mod 7$. Find r.

Step 1: Recall known facts: $5^6 \equiv 1 \mod 7$

Step 2: Manipulate the numbers using known facts and rules:

$5^{600} \equiv 5^{6*100} \equiv (5^6)^{100} \equiv 1^{100} \equiv 1 \mod 7$