

**Math 162 Chapter 11/Sections 1-8 Topic: Series and Sequences Worksheet**

**1. Sequence.**

- a. Define what a sequence is.
- b. Define Increasing and decreasing sequences/ Monotonic sequences
- c. State the Monotonic Sequence Theorem
- d. List the fibonacci sequences
- e. Give an example of a Recurrence Relations
- f. Practice Problems:

a. List first 5 terms for the following sequences

1.  $a_n = \frac{2^n}{2n+1}$

2.  $a_n = \frac{1}{n!}$

3.  $a_n = 5n$

4.  $a_1 = 1, a_{n+1} = 3a_n + 2$

5.  $a_n = \frac{(-1)^n}{n}$

b. Find a general formula for the following sequences.

1.  $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \dots$

2. 1,3,5,7,9,.....

3. 3,6,9,12,15, .....

4. 1,0,-1,0,1,0,.....

5.  $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{16}, \dots$

c. Determine whether if the following sequence converges or diverges, if converges then find the limit.

1.  $a_n = \frac{3 + 5n^2}{n + n^2}$

2.  $a_n = \frac{3n^2}{n+7}$

3.  $a_n = 4 + (0.76)^n$

## 2. Series.

- a. Define Infinite Series
  
  
  
  
  
  
  
  
  
  
- b. Partial Sums
  
  
  
  
  
  
  
  
  
  
- c. Give an example of an Arithmetic Series
  
  
  
  
  
  
  
  
  
  
- d. Give an example of a Geometric Series
  
  
  
  
  
  
  
  
  
  
- e. Give an example of a Telescoping Series

f. Give an example of a Harmonic Series

g. Test for Divergence (Theorem)

h. Practice Problems:

a. Find the first 5 terms of the following series

1.  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

2.  $\sum_{n=1}^{\infty} \frac{1}{n}$

3.  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

b. Check whether this series converges/diverges. If the series converges then find the partial sum.

1.  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

2.  $\sum_{n=1}^{\infty} \frac{1}{n}$

3.  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

### 3. Integral Tests

a. Define the Integral Test.

b. Practice Problems:

Use the integral test to determine whether this series converges or diverges

1.  $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$

2.  $\sum_{n=1}^{\infty} \frac{1}{n^2+n^3}$

3.  $\sum_{n=1}^{\infty} \frac{n^2}{n^3+4}$

#### 4. Comparison tests

a. Define what a comparison test is.

b. Define the Limit Comparison test

c. Practice Problems:

Use the comparison test or the Limit comparison test to determine whether the series converges/diverges. If the series converges then find the partial sum.

1.  $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

2.  $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$

3.  $\sum_{n=1}^{\infty} \frac{1}{12n+5}$

**5. Alternating Series**

a. Define what alternating series is.

b. Practice Problems:

Determine whether the series converges or diverges.

1.  $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \dots$

2.  $\sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{1}{n^2+4} \right)$

3.  $\frac{1}{\ln(3)} - \frac{1}{\ln(4)} + \frac{1}{\ln(5)} - \frac{1}{\ln(6)} + \dots$

**6. Absolute Convergence and the Ratio and Roots Tests**

a. Define the following terms:

1. Absolute Convergence

2. Ratio Test

3. Root Test

b. Practice Problems:

Determine whether this series converges by using absolute, ratio or root test.

1.  $\sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{7n+2} \right)$

2.  $\sum_{n=1}^{\infty} \frac{n}{5^n}$

3.  $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

**7. Power Series**

a. Define the Power Series.

b. Practice Problems:

Find the radius of convergence or interval of convergence.

1.  $\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$

**Solutions**

**1. Sequence.**

- a. Define what a sequence is.

A sequence is a list of number which can be written in a definite order,

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

$a_1$  is called the first term,  $a_2$  is called the second term and  $a_n$  is called the last term.

- b. Define Increasing and decreasing sequences/ Monotonic sequences

A sequence  $\{a_n\}$  is called an increasing sequence when  $a_n \leq a_{n+1}$  when  $n \geq 1$ . For example  $a_1 < a_2 < \dots < a_n$ . A sequence is decreasing when  $a_n \geq a_{n+1}$ . When a sequence is either increasing or decreasing it is monotonic.

- c. State the Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

- d. List the fibonacci sequences

$$1, 1, 2, 3, 5, 8, 13, 21, \dots$$

- e. Give an example of a Recurrence Relations

You can have multiple answers. Examples: Fibonacci Sequence, Tower of Hanoi, etc.

- f. Practice Problems:

- a. List first 5 terms for the following sequences

1.  $a_n = \frac{2^n}{2n+1}$

$$\frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \frac{16}{9}, \frac{32}{11}$$

2.  $a_n = \frac{1}{n!}$

$$1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}$$

3.  $a_n = 5n$

$$5, 10, 15, 20, 25$$

4.  $a_1 = 1, a_{n+1} = 3a_n + 2$

$$1, 5, 17, 53, 161$$

5.  $a_n = \frac{(-1)^n}{n}$

$$-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}$$

b. Find a general formula for the following sequences.

1.  $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \dots$

$$a_n = \frac{1}{2n}$$

2. 1, 3, 5, 7, 9, .....

$$a_n = 2n + 1$$

3. 3, 6, 9, 12, 15, .....

$$a_n = 3n$$

4. 1, 0, -1, 0, 1, 0, .....

$$a_n = \cos\left(\frac{n\pi}{2}\right)$$

5.  $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{16}, \dots$

$$a_n = \frac{1}{2^n}$$

c. Determine whether if the following sequence converges or diverges, if converges then find the limit.

1.  $a_n = \frac{3 + 5n^2}{n + n^2}$

$$\lim_{n \rightarrow \infty} \frac{10n}{1+2n} = \lim_{n \rightarrow \infty} \frac{10}{2} \text{ by L'Hospital Rule. } \lim_{n \rightarrow \infty} \frac{10}{2} = 5. \text{ Hence this sequence converges.}$$

2.  $a_n = \frac{3n^2}{n+7}$

$$\lim_{n \rightarrow \infty} \frac{3n^2}{n+7} = \lim_{n \rightarrow \infty} 6n = \infty. \text{ Hence this sequence diverges by L'Hospital rule.}$$

3.  $a_n = 4 + (0.76)^n$

$$\lim_{n \rightarrow \infty} 4 + (0.76)^n = \lim_{n \rightarrow \infty} 4 + 0 = 4. \text{ Thus the sequence converges.}$$



2. Series.

- a. Define Infinite Series

A sum of infinite sequences  $\{a_n\}_{n=1}^{\infty}$  which is expressed in a form as

$a_1 + a_2 + a_3 + \dots + a_n + \dots$ . This is called an infinite series. Or simply it can be denoted in a form  $\sum_{n=1}^{\infty} a_n$ .

- b. Partial Sums

The sum up to the  $n^{\text{th}}$  term

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

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$$s_n = a_1 + a_2 + \dots + a_n$$

- c. Give an example of an Arithmetic Series

$$\sum_{k=1}^{\infty} 2k = 2 + 4 + 6 + 8 + \dots + 2k + \dots$$

- d. Give an example of a Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots + ar^{n-1} + \dots \quad \text{where } a \neq 0 \text{ and } r \text{ is a common ratio.}$$

$$s_n = \frac{a(1-r^n)}{1-r} \quad \text{only if } -1 < r < 1.$$

- e. Give an example of a Telescoping Series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \dots + \left( \frac{1}{n} - \frac{1}{n+1} \right) = \lim_{n \rightarrow \infty} 1 - \frac{1}{n+1} = 1 - 0 = 1.$$

- f. Give an example of a Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}. \quad \lim_{n \rightarrow \infty} \frac{1}{n} = 0, \text{ but } \sum_{n=1}^{\infty} \frac{1}{n} \text{ does not converge.}$$

- g. Test for Divergence (Theorem)

If a limit does not exist or approaches infinity or  $\lim_{n \rightarrow \infty} a_n \neq 0$ . Then  $\sum_{n=1}^{\infty} a_n$  diverges.

h. Practice Problems:

a. Find the first 5 terms of the following series

$$1. \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$1, \frac{5}{4}, \frac{49}{36}, \frac{205}{144}, \frac{5269}{3600}$$

$$2. \sum_{n=1}^{\infty} \frac{1}{n}$$

$$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}$$

$$3. \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$$

b. Check whether this series converges/diverges. If the series converges then find the partial sum.

$$1. \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$\frac{\pi^2}{6}$ . Use a calculator and sum up all the terms. Once you reach a larger number you will get closer and closer to  $\frac{\pi^2}{6}$ .

$$2. \sum_{n=1}^{\infty} \frac{1}{n}$$

Since we have an harmonic series this series diverges.

$$3. \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \dots + \left( \frac{1}{n} - \frac{1}{n+1} \right) = \lim_{n \rightarrow \infty} 1 - \frac{1}{n+1} = 1 - 0 = 1. \text{ Since we reach a finite limit, the series converges.}$$

### 3. Integral Tests

- a. Define the Integral Test.

$f$  must be continuous, positive, decreasing function on  $[1, \infty)$ .

If  $\int_1^{\infty} f(x) dx$  is convergent then  $\sum_{n=1}^{\infty} a_n$  is convergent. If  $\int_1^{\infty} f(x) dx$  is divergent then  $\sum_{n=1}^{\infty} a_n$  is divergent.

- b. Practice Problems:

Use the integral test to determine whether this series converges or diverges

1.  $\sum_{n=1}^{\infty} \frac{1}{n^2+3} = \int_0^{\infty} \frac{1}{x^2+3} dx = \lim_{t \rightarrow \infty} \int_0^t \frac{1}{x^2+3} dx. x = u\sqrt{3}. dx = \sqrt{3}du.$

$$\int_0^t \frac{\sqrt{3}}{(u\sqrt{3})^2+3} du = \int_0^t \frac{\sqrt{3}}{3(u^2+1)} du = \frac{1}{\sqrt{3}} \arctan(u) = \frac{1}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) = \frac{\pi}{2\sqrt{3}}. \text{ So by integral test, this series converges.}$$

2.  $\sum_{n=1}^{\infty} \frac{1}{n^2+n^3} = \int_0^{\infty} \frac{1}{x^2+x^3} dx = \int_0^{\infty} \frac{1}{x^2} + \frac{1}{x+1} - \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_0^t \frac{1}{x^2} + \frac{1}{x+1} - \frac{1}{x} dx = \frac{-1}{x} + \ln(x+1) - \ln(x) = \infty.$

Therefore this series diverges by integral test.

3.  $\sum_{n=1}^{\infty} \frac{n^2}{n^3+4} = \int_0^{\infty} \frac{x^2}{x^3+4} dx = \lim_{t \rightarrow \infty} \int_0^t \frac{x^2}{x^3+4} dx. u = x^3 + 4. du = 3x^2 dx. \frac{1}{3} du = x^2 dx.$

$$\frac{1}{3} \ln(u) = \frac{1}{3} \ln(x^3 + 4) = \infty. \text{ Hence the series diverges by integral test.}$$

### 4. Comparison tests

- a. Define what a comparison test is.

For all  $n \in \mathbb{N}$ , when  $a_n \leq b_n$ , if  $\sum b_n$  is convergent then  $\sum a_n$  is also convergent.

For all  $n \in \mathbb{N}$ , when  $a_n \geq b_n$ , if  $\sum b_n$  is divergent then  $\sum a_n$  is also divergent.

b. Define the Limit Comparison test

If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$  whereas  $c > 0$  and  $c$  is a finite number then  $\sum a_n$  and  $\sum b_n$  converges or  $\sum a_n$  and  $\sum b_n$  diverges.

c. Practice Problems:

Use the comparison test or the Limit comparison test to determine whether the series converges/diverges. If the series converges then find the partial sum.

1.  $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

Since  $\frac{1}{n^2+4} < \frac{1}{n^2}$  and  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges, then by comparison test this series converges.

2.  $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$

Since  $\frac{1}{n^2+5} < \frac{1}{n^2}$  and  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges, then by comparison test this series converges.

3.  $\sum_{n=1}^{\infty} \frac{1}{12n+5}$

Observe that  $\frac{1}{12n+5} > \frac{1}{12n+12}$  because  $12n+12 > 12n+5$ . So  $\sum_{n=1}^{\infty} \frac{1}{12n+12} = \frac{1}{12} \sum_{n=1}^{\infty} \frac{1}{n+1}$ . So  $\sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=2}^{\infty} \frac{1}{n}$ . Since the  $\sum_{n=2}^{\infty} \frac{1}{n}$  is a harmonic series, it diverges. So by the comparison test, this series diverges.

## 5. Alternating Series

a. Define what alternating series is.

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

This only satisfies when for all  $n \in \mathbb{N}$ ,

$$a_{n+1} \leq a_n \text{ and } \lim_{n \rightarrow \infty} a_n = 0.$$

b. Practice Problems:

Determine whether the series converges or diverges.

$$1. \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \dots$$

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n+1}$ .  $\lim_{n \rightarrow \infty} \frac{1}{2n+1} = \frac{1}{\infty} = 0$ . Therefore by Alternating Series Test, this series converges.

$$2. \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{1}{n^2+4} \right)$$

$\lim_{n \rightarrow \infty} \frac{n}{n^2+4} = \lim_{n \rightarrow \infty} \frac{1}{2n}$  by L'Hospital Rule.  $\lim_{n \rightarrow \infty} \frac{1}{2n} = 0$ . Therefore this series converges by Alternating Series Test.

$$3. \frac{1}{\ln(3)} - \frac{1}{\ln(4)} + \frac{1}{\ln(5)} - \frac{1}{\ln(6)} + \dots$$

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\ln(n+2)}$ .  $\lim_{n \rightarrow \infty} \frac{1}{\ln(n+2)} = 0$ . Therefore this series converges by Alternating Series Test.

## 6. Absolute Convergence and the Ratio and Roots Tests

a. Define the following terms:

1. Absolute Convergence

If  $\sum |a_n|$  is convergent then  $\sum a_n$  is convergent.

2. Ratio Test

If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ , then the series  $\sum a_n$  is absolutely convergent.

If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ , then the series  $\sum a_n$  diverges.

If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ , then the Ratio Test is Inconclusive.

3. Root Test

If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$ , then the Root Test is inconclusive.

If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$ , then the series  $\sum a_n$  is absolutely convergent.

If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$  or  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$ , then the series  $\sum a_n$  is divergent.

b. Practice Problems:

Determine whether this series converges by using absolute, ratio or root test.

$$1. \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{7n+2}\right)$$

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{7n+2} \right| = \sum_{n=1}^{\infty} \frac{1}{7n+2}$ . So  $\frac{1}{7n+2} > \frac{1}{7n+7} = \frac{1}{7} \left(\frac{1}{n+1}\right)$ . So since  $\frac{1}{n}$  is a harmonic series, the series diverges.

$$2. \sum_{n=1}^{\infty} \frac{n}{5^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)}{5^{(n+1)}}}{\frac{n}{5^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+1}{5^{n+1}} * \frac{5^n}{n} \right| = \frac{5^n}{5^n * 5} \lim_{n \rightarrow \infty} \left| \frac{n+1}{n} \right| = \frac{1}{5} \lim_{n \rightarrow \infty} \frac{n+1}{n} = \frac{1}{5} * 1 = \frac{1}{5} < 1.$$

$$3. \sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$$

$\lim_{n \rightarrow \infty} \sqrt[5n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{2n}{n+1} = \lim_{n \rightarrow \infty} 2 = 2$ . Since  $2 > 1$  this series diverges by root test.

7. Power Series

a. Define the Power Series.

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots$$

b. Practice Problems:

Find the radius of convergence or interval of convergence.

$$1. \sum_{n=1}^{\infty} \frac{x^{2n}}{n!} = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{2(n+1)}}{(n+1)!}}{\frac{x^{2n}}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(n+1)(n)!} * \frac{n!}{x^{2n}} \right| = x^2 \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 < 1.$$

Therefore the Radius of convergence is  $-\infty < x < \infty$ .